DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA 91125

Permits or Taxes? How to Regulate Cournot Duopoly with Polluting Firms

Till Requate
California Institute of Technology
University of Bielefeld, Germany

SOCIAL SCIENCE WORKING PAPER 792

March 1992

Permits or Taxes? How to Regulate Cournot Duopoly with polluting firms.

by Till Requate,
Institute of Mathematical Economics,
University of Bielefeld, Germany

Abstract

The paper investigates pollution control of firms engaging in imperfect competition. We consider asymmetric Cournot duopoly where firms have linear technologies. Welfare is assumed to be separable in consumers' surplus and social damage which is given by a convex function depending on the aggregate pollution level. After deriving social optimum, we give a complete characterization of the optimal linear tax as well as of the optimal number of permits taking into account the firms' strategic behavior, and then compare the two both policies with respect to welfare. None of them turns out to implement social optimum in general. Also, no policy can be said to be superior. for all parameters. However, for a considerable range of parameters giving out permits yields a higher welfare than taxes. Finally, we consider double taxation of output *and* pollutants. In this case social optimum can always be achieved, if there are only two firms.

Permits or Taxes? How to Regulate Cournot Duopoly with Polluting Firms

Till Requate*

1 Introduction

In order to reduce the aggregate output of pollutants, in practice, most governments or jurisdictions just impose uniform standards of pollution levels on the firms. From economic theory we know for almost 30 years that there are more efficient tools than that, like charging Pigouvian taxes on pollutants or to give out a number of marketable permits for pollutants. It is well known that under certain assumptions both regimes are equivalent and yield the socially efficient outcome if the optimal Pigouvian tax is charged or the optimal number of permits is given out (for an exposition see for example BAUMOL and OATES [2]). For this results to hold it is necessary to assume that the government has complete information about the industry's aggregate abatement cost as well as about the social damage which is assumed to depend only on aggregate emissions. Moreover, it has to be assumed that the polluting firms supply their marketable output on a competitive market and *also* behave as price takers on the market for permits if there is any. SPULBER [14] demonstrates that under these assumptions taxes or permits are also optimal in the long run if the number of firms is determined endogenously.

Under incomplete information about aggregate abatement cost and damage, either of both tools can yield a higher welfare, contingent on the ratio of slopes of marginal abatement cost and marginal social damage as Weitzman showed. Further approaches on incomplete information have been pursued by ROBERTS and SPENCE [11] and also by KWEREL [7], who propose a mixture of measures consisting of permits, taxes and

*University of Bielefeld, Germany. This paper has been written for the most part during a visit to California Institute of Technology, Pasadena, California. The author would like to express his gratitude to the Division of the Humanities and Social Sciences for its hospitality, to P. Chander and the participants of seminars in Pasadena and Bielefeld for their helpful comments, and especially to Jeanne Netzley for her TEX–nical support.

subsidies on abatement. All these models are partial analyses where firms are assumed to behave as price takers.[1]

Very few has been worked on how to regulate polluting firms which engage in imperfect competition, that is, have market power on either market. HAHN [6] studies a model where one big firm has market power on the market of permits, the remaining firms behave as price takers. He shows that the final allocation of permits depends on the initial allocation and will be inefficient in general. MALUEG [8] considers the distribution of permits in a Cournot oligopoly on the output market, however, without explicitly considering the "pollution technology". In a recent paper, EBERT [4] investigates taxation of a *symmetric* Cournot oligopoly with polluting firms. In that special case, social optimum can be implemented by a suitable Pigouvian tax. However, the analysis of imperfect competition starts to become interesting if firms have *different* technologies, otherwise uniform standards would work quite all right.

According to my knowledge, asymmetric oligopoly models, where the output market as well as the pollution sector —i.e. the market for permits if there is any — are treated simultaneously have not been analyzed so far. The reason may be that probably few can be said under fairly general assumptions. Hence, in this paper, an asymmetric duopoly model with special, but not unrealistic technologies will be set up. Each of two firms owns a linear or Leontief technology, that is, it faces constant marginal cost and produces a pollutant proportional to the output of the marketable commodity. We pursue partial analysis for one marketable commodity and assume that a certain pollutant will be generated by this industry only. (Partial) social welfare is additively separable in consumers' surplus, social damage of pollutant and production cost. The government has complete information about market demand, social damage and about the two technologies, but does not necessarily know what firm owns what technology. After deriving the social optimum, we give a complete characterization of the optimal linear tax as well as the optimal number of permits, taking into account the firms' strategic behavior on the output market as well as on the market for permits if there is one. It will turn out that the optimal (linear) tax will be nondecreasing as a function of a damage parameter s, which determines the slope of marginal damage. For low s the Pigouvian tax will be negative. So, if social damage from pollution is low, pollution will be subsidized in order to increase output, a result in accordance with [4]. The optimal number of permits, on the other hand, is nonincreasing in s, downward jumping for some value of s, and constant on some interval for s. Both regimes do in general *not* implement the first best solution if firms are different, especially not when both firms are active. Moreover, none of the two policies can be said to be superior to the other, in terms of welfare. Both policies may yield social optimum, allowing only the less polluting firm to produce, if social damage from pollution is sufficiently high. Under the permit regime, however, the social optimum is achieved for a greater range of parameters for which it is socially desirable that only the "cleaner" firm produces. This yields an argument in favor of permits under imperfect

[1]There is much more literature on taxes and permits under price taking behavior, which we cannot all give credit here. For an excellent overview of different kinds of permit trading see TIETENBERG's book [17].

competition if social damage is high. For very low social damage from pollution, the permit regime turns out to be undesirable since the lower cost firm exploits the regime by buying all the permits and exercises monopoly power. For intermediate values of the social damage parameter, few can be said in general. Recently, H. SIEBERT, president of the *Institut fuer Weltwirtschaft*, argued in a magazine's interview in favor of permits by explaining its idea by the example of power plants: the modern power plant buys all the permits from the odd one, compensating it for closing down. This paper supports his argument in the case that the odd firm is sufficiently worse or social damage is sufficiently high (and the corresponding number of permits is low), however, not for *all* values of s for which it is *socially desirable* that the worse polluter closes down!

In this duopoly model, the inefficiency arising from the two policies, can be removed by taxing both, output *and* pollutants. In that case, it is easy to show that social optimum can always be implemented. This result, however, does certainly not generalize to nonlinear technologies and to more than two firms.

The paper is organized as follows: In the following section we set up the model. Section 3 characterizes the social optimum. In section 4 we briefly discuss the underlying information structure for the tax and the permit regime. In sections 5 and 6 we develop the optimal linear tax, and the optimal number of permits, respectively. In section 7 we compare the two regimes and give a numerical example. Section 8 deals with double taxation of output and emissions. The last section concludes. Unless stated otherwise all formal proofs are given in the appendix.

2 The Basic Model

Throughout this paper we will consider a Cournot duopoly with firms $i = 1, 2$ setting quantities q_1, q_2. The price is determined by an inverse demand function P, with $P' < 0$, which depends on aggregate output $Q = q_1 + q_2$. We assume there is a finite choke–off price $\overline{p} := P(0) := \min\{p | D(p) := P^{-1}(p) = 0\}$. For various reasons, we further make

Assumption 1 *$|P''|$ is sufficiently bounded; in particular: for all $Q > 0$: $P''(Q) < 2P'(Q)/Q$.*

The upper bound for P'' is sufficient to guarantee the second order conditions for profit maximization of monopoly as well as for the duopolistic firm in Cournot–Nash equilibrium. It is also sufficient to guarantee uniqueness of Nash–equilibrium.[2]

Both firms have constant marginal costs c_1 and c_2, with (w.l.o.g.) $c_1 \leq c_2 < \overline{p}$. Production is not possible without pollution. Producing q_i units of output, firm i pro-

[2]Later on, we also need that P is not too concave. To quantify the lower bound, however, yields tedious expressions and does not yield further insight.

duces $e_i = d_i q_i$ units of emissions. This cost and pollution structure may be considered stemming from a linear (Leontief–)technology. Firms do not have an abatement technology.[3] Total emissions are written $E := e_1 + e_2$. To evaluate utility and harm of (q_1, q_2) (which determines (e_1, e_2)) to the society, we assume to have a partial social welfare function $W(q_1, q_2)$.[4] In the absence of pollution, in the industrial economics literature, a social welfare is simply taken as $W(q_1, q_2) = \int_0^Q P(z)dz - c_1 q_1 - c_2 q_2$, that is, consumers' gross surplus minus aggregate production costs.[5] We will extend this approach by assuming that benefit from production and damage from pollution are additively separable. This means, in addition to consumers' surplus there is a social damage function $S : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ with $(E, s) \mapsto S(E, s)$, which depends on aggregate emissions E and a damage parameter s. Employing the usual notation $S_1(E, s) := \frac{\partial S(E,s)}{\partial E}$ and so on, we make the following assumption.

Assumption 2 *o) S is at least twice continuously differentiable with respect to[6] E and s; in $(0,0)$ the right sided partial derivatives exist.*
i) $S(0, s) = 0 \; \forall s \geq 0$,
ii) $S(E, 0) = 0 \; \forall E \geq 0$,
iii) $S_1(E, s) \geq 0 \; \forall s > 0$ and strictly greater for $E > 0$.
iv) $S_{11}(E, s) \geq 0 \; \forall s > 0$ and strictly greater for $E > 0$.
v) $S_{12}(E, s) > 0 \; \forall E > 0, s > 0$.

So, S is increasing and convex in E and marginal damage increases in s. Although s is an exogenous parameter of the model, parameterizing S via s allows us to characterize social optimum and also regulatory policies as a function of the damage function's steepness. Finally we assume:

Assumption 3 *The pollutant resulting from production of the industry's output, only arises in* this *industry.*

Assumption 3 does not hold in *all* industries, of course. For example CO_2, is generated by many different industries. SO_2, on the other hand, is generated basically by power plants. Also in the chemical industry, some poisonous pollutants are generated from

[3]Assuming the firms to have an abatement technology allowing them to reduce pollution by investing some effort, which in turn generates a higher cost, screws up the linearity of the cost function. Then we could start immediately with some cost function $C(q, e)$ which is nonlinear in output and emissions. This is certainly worth to pursue and should be tackled by further research. However, much less can be derived in general, as far as I can see.

[4]We call it *partial* since we neglect income effects and externalities on *other* markets.

[5]This is equivalent to $W(q_1, q_2) = \int_0^Q P(z)dz - P(Q) \cdot Q + (P(Q)q_1 - c_1 q_1) + (P(Q)q_2 - c_2 q_2)$, that is, net consumers' surplus plus profits of the firms. Some authors use the latter, and sometimes even multiply surplus and profits with different weights (see for example BARON and MEYERSON [1]). Then, however, the two concepts are not equivalent.

[6]For short: "w.r.t." in the remainder.

production of one certain commodity. Since we want to analyze regulation of firms under imperfect competition, Assumption 3 is crucial to make the analysis interesting.

Assuming separability of social welfare in consumers' surplus, production cost, and social damage, the welfare function is given by

$$W_s(q_1, q_2) := \int_0^Q P(z)dz - S(E, s) - c_1 q_1 - c_2 q_2 \qquad (2.1)$$

Without any kind of regulation, Cournot competition leads to a Cournot–Nash equilibrium independently of s. By Assumption 1 there is always a unique equilibrium for all constant marginal costs $c_1, c_2 < \overline{p}$.

Before turning to regulatory policies, let us derive the social optimum a fictive social planner would install under complete information. If $c_1 < c_2$, it is clear that for $s = 0$ the higher cost firm 2 should not produce anything. If social damage is very high, one could think that only the firm with the relatively lower pollution level per unit of output should operate, that is, with the smaller d_i. However, it is not quite like this. What will turn out to be crucial is whether the term $(d_1 c_2 - d_2 c_1)/(d_1 - d_2)$ is greater than the choke–off price or not, or equivalently, what the sign is of $d_1/(\overline{p} - c_1) - d_2/(\overline{p} - c_2)$, which is the difference between the firms' ratio of marginal pollution and maximal marginal consumers' surplus. For convenience, we write for short $\Delta := d_1(\overline{p} - c_2) - d_2(\overline{p} - c_1)$ for the remainder of the paper.

3 The social optimum

The social planner has to solve the following program:

$$\max_{q_1, q_2} W_s(q_1, q_2) := \max_{q_1, q_2} \int_0^{q_1+q_2} P(z)dz - S(d_1 q_1 + d_2 q_2, s) - c_1 q_1 - c_2 q_2 \qquad (3.1)$$

s.t. $q_1 \geq 0$, $q_2 \geq 0$.

The following proposition yields the properties of the optimal solution (remember that we assumed $c_1 \leq c_2$):

Proposition 3.1 *a) If $\Delta \leq 0$, firm 2 never produces for all $s \geq 0$, unless $c_1 = c_2$ $d_1 = d_2$, and firm 1 produces q which solves*

$$P(q) = c_1 + S_1(d_1 q, s) d_1 \ . \qquad (3.2)$$

q is decreasing in s. (If both firms are alike, clearly q may be arbitrarily distributed on both firms).

b) If $\Delta > 0$, there are parameters $\underline{s} < \infty$ and $\overline{s} \leq \infty$ ($< \infty$ for $d_2 > 0$) with $\underline{s} < \overline{s}$, such that the solution of (3.1) is characterized by

$$\left.\begin{array}{l} q_1 > 0 \\ q_2 = 0 \end{array}\right\} \quad \forall \quad 0 \leq s \leq \underline{s} \, ,$$

and $Q = q_1$ is decreasing in s.

$$\left.\begin{array}{l} q_1 > 0 \\ q_2 > 0 \end{array}\right\} \quad \forall \quad \underline{s} < s < \overline{s} \, ,$$

and q_1 is decreasing, q_2 is increasing, and $Q = q_1 + q_2$ is constant in s.

$$\left.\begin{array}{l} q_1 = 0 \\ q_2 > 0 \end{array}\right\} \quad \forall \quad s \geq \overline{s} \text{ if } d_2 > 0,$$

and $Q = q_2$ is decreasing in s.

Moreover, Q, E, and W are continuous, E and W are decreasing in s.

Thus, we can say that firm one has the better technology if $\Delta \leq 0$, unless $c_1 = c_2$, $d_1 = d_2$ when production can be arbitrarily shared by both firms. Notice that $c_1 < c_2$ and $d_1 = d_2$ as well as $c_1 = c_2$ and $d_1 < d_2$ imply $\Delta \leq 0$. But notice also that $\Delta \leq 0$ may hold for some $d_1 > d_2$ if c_1 is sufficiently smaller than c_2. In other words, even if firm 2 emits less pollutants per unit of output, it may never produce in social optimum if the cost differential $c_2 - c_1$ is sufficiently high.

Proposition 3.1 is derived by solving (3.1), taking into account the Kuhn–Tucker conditions with respect to the constraints $q_1 \geq 0$ and $q_2 \geq 0$. Details are relegated to the appendix. Notice that $c_1 \leq c_2$ and $\Delta > 0$ imply $d_1 > d_2$, that is, firm 2 emits strictly less pollutants per unit of output than firm 1. Interestingly, for $\underline{s} \leq s \leq \overline{s}$ aggregate output is constant in s and equals $Q = \Delta/(d_1 - d_2)$. Thus, the social planner shifts production continuously from firm 1 to firm 2 as s increases, keeping total output constant, until firm 1, which faces the lower production cost but is the worse polluter, shuts down. These properties are displayed in figure 1.

Figure 1 about here.

4 Regulatory Policies: Some Remarks on the Information Structure

Needless to say that first best solutions are in general not enforceable by prescribing the firms to produce individually different quantities. Not only is there an information problem in the sense that the government does not know the firms' technologies. It is also considered to be unfair to prescribe different policies to the firms. By widespread opinion of the public and their representatives, firms are supposed to make their own decisions about their output in an economy with free enterprise. This paper is not about incomplete information in the sense that the government has prior (probability) beliefs about the firms' technologies. If the government, however, has to choose a "fair" policy that treats all the firms alike, complete information is not necessary anyway. To choose, for instance, an optimal linear tax, it is sufficient to know the existing types of technologies and how many there are of each type, but not exactly, what firm has what technology.[7] Hence, we will assume for the remainder of this paper that the government knows at least what technologies there are.

We also assume that the emissions generated by each firm can be perfectly monitored by the authorities without costs. So, the firms will pay a tax bill exactly according to the amount of their emitted pollutants (in section 5). In case of holding permits, firms cannot emit more than the number of permits allows them to do. Otherwise, we assume, a high penalty has to be paid. So there is no room for moral hazard. Needless to say that also this a strong abstraction.

5 Pigouvian Taxes

By a Pigouvian Taxes we mean a linear tax tariff on emissions. Firm i has to pay a bill of $\tau \cdot e_i$ if it emits e_i units of the pollutant, where τ is the tax rate. Producing q_i units, firm i's costs amount to $c_i q_i + \tau e_i = (c_i + \tau d_i) q_i$. We do not impose a condition on the sign of τ. Negative τ's, mean a subsidy. Indeed, we will see that for low social damage it is optimal to subsidize pollution, a seemingly perverse phenomenon at first thought. Since we retain the assumption of Cournot competition, the firms go on choosing Cournot–Nash quantities if τ is such that it is profitable for both firms to produce, and firm i produces its monopoly output if firm j chooses $q_j(\tau) = 0$. This behavior can be gathered in the following equation.

$$P(q_i(\tau) + q_j(\tau)) + P'(q_i(\tau) + q_j(\tau))q_i(\tau) - (c_i + \tau d_i) = 0 \tag{5.1}$$

$\forall i$ with $q_i(\tau) > 0$ and $\forall j \neq i$ with $q_j(\tau) \geq 0$.

What is the government's program? It wants to find the optimal tax rate under the constraint that the firms set Nash quantities if they both produce, and monopoly quantities if only one of them is active, that is, if $q_i(\tau)$ is given by (5.1). Hence, it has to

[7]This information structure is reminiscent of the second degree price discrimination literature. In Maskin and Riley's model [9], the monopolist has to know what kinds of consumer there are, but not which consumers has which utility function. The same structure can be found in Rothschild and Stiglitz [12], and Stiglitz [16] in the analysis of insurance markets. Of course, assuming this information structure is more appealing when there are many agents rather than only two as in our model.

solve $\max_\tau W_s^{PT}(\tau)$

$$:= \max_\tau \int_0^{q_1(\tau)+q_2(\tau)} P(z)dz - S(d_1 q_1(\tau) + d_2 q_2(\tau), s) - c_1 q_1(\tau) - c_2 q_2(\tau) \qquad (5.2)$$

Observe that the additional costs of size τq_i for the firms and the tax revenue for the government cancel out if we assume that the government redistributes them lump sum back to the firms, or even to consumers. This does not matter. What matters is that the government has no objective to collect tax revenues in this industrial sector. Especially, there is no additional technology the government can buy in order to reduce the aggregate emissions E, once these have been dumped into the environment by the firms. To solve (5.2), it is useful to know the behavior of $q_i(\tau)$, $i = 1, 2$, especially, what firm closes first and when the other firm switches to monopoly behavior as τ increases. Let τ_i^D be the duopoly tax (or subsidy), at which firm i just closes in competition with firm j, that is, τ_i^D satisfies

$$q^i(\tau_i^D) = 0 \ , \quad q^i(\tau) > 0 \text{ for } \tau < \tau_i^D \text{ or } \tau > \tau_i^D \ , \text{ and } q_j(\tau_i^D) > 0 \ . \qquad (5.3)$$

Lemma 5.1 *a) If $\Delta < 0$, firm 2 does not produce at all $\forall\tau$, or closes first as τ increases, formally the latter means, $\exists\tau_2^D$ such that $q_2(\tau) = 0 \ \forall\tau \geq \tau_2^D$, $q_2(\tau) > 0 \ \forall\tau < \tau_2^D$ and $q_1(\tau_2^D) > 0$.*

b) If $\Delta = 0$, firm 2 does not produce at all $\forall\tau$, or both firms close simultaneously.

c) If $\Delta > 0$, firm 1 closes first as τ increases, formally the latter means, $\exists\tau_1^D$ such that $q_1(\tau) = 0 \ \forall\tau \geq \tau_1^D$, $q_1(\tau) > 0 \ \forall\tau < \tau_1^D$ and $q_2(\tau_1^D) > 0$.

The next Lemma converts Lemma 5.1 c)

Lemma 5.2 *If $\exists\tau_1^D$ satisfying (5.3) then $\Delta > 0$.*

Let $\tau(s) := \arg\max_\tau W_s^{PT}(\tau)$ be the optimal linear emission tax, and let $S^D := \{s \in \mathbb{R} | q_i(\tau(s)) > 0 \text{ for } i = 1, 2\}$ be the set of those damage parameters where both firms produce under the optimal tax, and let $\overline{S}^D$ be its closure. Let $\tau^D(s) := \tau(s)$ for those s that are in $\overline{S}^D$. First order conditions[8] imply

$$\frac{dW_s^{PT}}{d\tau}(\tau^D(s)) = 0 \quad \forall s \in \overline{S}^D \qquad (5.4)$$

taking right/left derivatives on the boundary of $\overline{S}^D$. Assume that the second order condition

$$\frac{d^2 W_s^{PT}}{d\tau^2}(\tau^D(s)) < 0 \quad \forall s \in \overline{S}^D \ , \qquad (5.5)$$

[8]for short: f.o.c.s for short in the remainder.

is satisfied. It can be shown that this is the case under Assumption 1.[9]

Lemma 5.3 *Under Assumption 1, i) $Q'(\tau) < 0$, ii) $E'(\tau) < 0$.*[10]

Differentiating (5.4) w.r.t. s and solving for the derivative $\tau^{D'}(s)$ yields

$$\tau^{D'}(s) = \frac{S_{12}(E(\tau), s)E'(\tau)}{\frac{d^2 W_s^{PT}}{d\tau^2}(\tau^D(s))} > 0 \tag{5.6}$$

by Assumptions 1 and 2 and Lemma 5.3. Hence, if there is $s \geq 0$ such that

$$\tau^D(s) = \tau_i^D \tag{5.7}$$

the solution is unique. Observe that in case of $\Delta > 0$, if solutions of (5.7) exist for both $i = 1, 2$, then $s_2^D < s_1^D$ by Lemma 5.2 and 5.3. Hence we define[11]

$$s_i^D := \begin{cases} \text{solution of (5.7) in } s & \text{if it exists,} \\ -\infty & \text{else if } i = 2 \\ \infty & \text{else} \end{cases} \tag{5.8}$$

This means, s_i^D is that damage parameter for which the value of tax function τ^D equals τ_i^D if such a parameter exists. The settings $-\infty$ and ∞ are made for convenience for later on.

For the subsequent analysis it is useful to consider briefly:

The case of pure monopoly. Let us assume for a moment that only firm j is around and is to be regulated by an emission tax.

Setting $q_i = 0$ in (5.1), we get the f.o.c. for profit maximization of the monopolistic firm j. Differentiating w.r.t. τ and solving for $q_j'(\tau)$ yields

$$q_j'(\tau) = \frac{d_j}{2P'(q_j) + P''(q_j)q_j} < 0 \tag{5.9}$$

by Assumption 1. The f.o.c. for the optimal tax implies

$$P(q_j(\tau)) - S_1(d_j q_j(\tau), s)d_j - c_j = 0 \; , \tag{5.10}$$

since $q_j'(\tau) \neq 0$. Let $\tau^{M_j}(s)$ be the optimal tax to regulate the monopolist j given the damage parameter s. Differentiating (5.10) w.r.t. s yields

$$\tau^{M_j'}(s) = \frac{S_{12}(d_j q_j(\tau), s)d_j^2}{[P'(q_j(\tau)) - S_{11}(d_j q_j(\tau), s)]q_j'(\tau)} > 0 \tag{5.11}$$

since $S_{12} > 0$, $P' < 0$, $S_{11} > 0$, $q_j' < 0$. As a byproduct we get

[9]Here we need the lower bound for P''. For linear demand, (5.5) is easily established.

[10]Also if you skip some proofs, you may look at this one and read Remark A.1.

[11]The superscript stands for "duopoly".

Corollary 5.1 *The optimal emission tax to regulate a monopolist yields social optimum.*

This follows simply from (5.10) which is also the f.o.c. of social optimum if only one firm were around. Details are omitted.

Back to duopoly. If there is $s \geq 0$ such that

$$\tau^{M_j}(s) = \tau_i^D \tag{5.12}$$

the solution is unique, since $\tau^{M_j}(s)$ is strictly increasing. Then let $\forall i = 1, 2,\ j = 3 - i$:

$$s_i^{M_j} = \begin{cases} \text{solution of (5.12) in } s & \text{if it exists,} \\ -\infty & \text{else if } i = 2 \\ \infty & \text{else} \end{cases} \tag{5.13}$$

This means, $s_i^{M_j}$ is that damage parameter for which the value of the monopoly tax function equals τ_i^D and where firm i would just be on the margin between opening and closing if the monopoly police τ^{M_j} applies.

The next Lemma is the keystone for the characterization of the optimal Pigouvian tax.

Lemma 5.4 *Let $W^{M_j}(\tau, s)$ be the welfare when only firm j is around and reacts as a monopolist upon the tax τ, and the damage parameter is s.*

a) If $\Delta < 0$ and $0 < s_2^D < \infty$, then

$$\frac{\partial W^{M_1}}{\partial \tau}(\tau_2^D, s_2^D) < 0 \ . \tag{5.14}$$

b) If $\Delta > 0$ and $0 < s_2^D < \infty$, then

$$\frac{\partial W^{M_1}}{\partial \tau}(\tau_2^D, s_2^D) > 0 \ . \tag{5.15}$$

c) If $\Delta > 0$ and $0 < s_1^D < \infty$, then

$$\frac{\partial W^{M_2}}{\partial \tau}(\tau_1^D, s_1^D) < 0 \ . \tag{5.16}$$

Basically, Lemma 5.4 says that, if the tax is such one firm, say i, is just on the margin to close down, whereas firm $j \neq i$ is still in the market, this tax rate is not optimal, if firm i were not around. The Lemma also indicates the directions into which the tax has to be moved in order to increase welfare. Lemma 5.4 implies:

Lemma 5.5 *a) If* $\Delta < 0$, *and* $s_2^D \geq 0$ *then* $s_2^D < s_2^{M_1}$. *Moreover, there are no* $\infty > s_1^{M_2}$, $s_1^D \geq 0$.

b) If $\Delta > 0$, *and* $s_2^{M_1} \geq 0$ *then* $s_2^{M_1} < s_2^D$.

After these preparations we are ready to characterize the optimal linear tax as a function of the damage parameter s.

Proposition 5.1
a) If $\Delta < 0$, *then*

$$\tau(s) = \begin{cases} \tau^D(s) & \text{for} \quad 0 \leq s \leq s_2^D, \quad \text{(both firms produce)} \\ \tau_2^D & \text{for} \quad \max\{0, s_2^D\} \leq s \leq s_2^{M_1}, \quad \text{(only firm 1 produces)} \\ \tau^{M_1}(s) & \text{for} \quad \max\{0, s_2^{M_1}\} \leq s \quad \text{(only firm 1 produces)} \end{cases} \tag{5.17}$$

b) If $\Delta > 0$, *then*

$$\tau(s) = \begin{cases} \tau^{M_1}(s) & \text{for} \quad 0 \leq s \leq s_2^{M_1} \quad \text{(only firm 1 produces)} \\ \tau_2^D & \text{for} \quad \max\{0, s_2^{M_1}\} \leq s \leq s_2^D, \quad \text{(only firm 1 produces)} \\ \tau^D(s) & \text{for} \quad \max\{0, s_2^D\} \leq s \leq s_1^D, \quad \text{(both firms produce)} \\ \tau_1^D & \text{for} \quad s_1^D \leq s \leq s_1^{M_2}, \quad \text{(only firm 2 produces)} \\ \tau^{M_2}(s) & \text{for} \quad s_1^{M_2} \leq s \quad \text{(only firm 2 produces)} \end{cases} \tag{5.18}$$

c) If $\Delta = 0$, *then*

$$\tau(s) = \tau^D(s) \qquad \forall\, s \geq 0 \qquad \textit{(both firms produce)} \qquad \textit{or} \tag{5.19}$$

$$\tau(s) = \tau^{M_1}(s) \qquad \forall\, s \geq 0 \qquad \textit{(only firm 1 produces)} \tag{5.20}$$

Proposition 5.1 follows immediately from Lemmata 5.1 – 5.5. Lemma 5.4 is most important among all and a bit tricky to prove. Notice that some of the intervals, for example $[0, s_2^{M_1}]$ may be empty.

In words, Proposition 5.1 says that if firm 2 has the strictly worse technology, that is if $\Delta < 0$, it *may* be the case that for low values of s *both* firms produce. By Lemma 5.1, firm 2 closes first as s increases. For $s \in [\max\{0, s_2^D\}, s_2^{M_1}]$, the tax is constant in s and equals τ_2^D. This is due an incentive constraint: Suppose $s_2^D > 0$. If s increases towards s_2^D, $\tau(s)$ goes to τ_2^D, that is, firm 2 closes down. For higher taxes than τ_2^D, firm 1 is a monopolist. Hence $\tau(s) \neq \tau^D(s)$, and firm 1 has to be taxed as a monopolist. However, if firm 2 could be prohibited to produce for s slightly higher than s_2^D, the optimal tax for the monopoly firm 1 would be lower than τ_2^D for $s \leq s_2^{M_1}$. This follows immediately from Lemma 5.4. But firm 1 cannot be told to shut down by law. At least this is what we assume. Hence, to prevent firm 1 from producing, the tax must not be lower than τ_2^D. For $s \geq s_2^{M_1}$, $\tau(s) = \tau^{M_1}(s) \geq \tau_2^D$, and $\tau(s)$ is strictly increasing in s. Notice that in case a) it can never happen that only firm 2 produces as a monopolist. This follows from the fact that firm 1 produces at least for $\tau = 0$.

In part b) of the proposition, where firm 1 has the lower cost $c_1 \leq c_2$, but firm 2 has the "cleaner" technology, it may be the case that the lower cost firm 1 produces as a monopolist for low damage parameters. Then, *both* firms produce for intermediate values of s, whereas for high s only the "cleaner" firm produces. Here, there may be two intervals for $\tau(s)$ being constant in s. On the first interval $[\max\{0, s_2^{M_1}\}, s_2^D]$ (which may be empty) we have $\tau(s) = \tau_2^D < 0$, that is, we get a subsidy.[12] On the second interval $[\bar{s}_1^D, \bar{s}_1^{M_2}]$ (which is always nonempty for $d_2 > 0$) we have $\tau(s) = \tau_1^D > 0$, that is, τ is a real tax. Depending on the parameters it is also possible that for $s = 0$, both firms produce under the optimal tax. But the case that firm 2 is a monopolist for all s is ruled out.

Figure 2 about here.

In figure 2 we have depicted the optimal tax as a function of s for the case b) of the proposition where all the $s_j^{M_i}$, s_j^D are positive.[13]

Corollary 5.2 *a) If $\Delta < 0$, the tax yields social optimum for $s \geq s_2^{M_1}$. If the firms are sufficiently different, in particular, if d_2 is sufficiently high, the tax solution yields social optimum $\forall s \geq 0$.*

b) If $\Delta > 0$, the tax yields social optimum for $s \in [0, s_2^{M_1}]$ and for $s \geq s_1^{M_2}$.

c) If $\Delta = 0$ and $c_1 \neq c_2$, the tax yields social optimum for no s, if for some s both firms produce under the optimal tax.

d) If $c_1 = c_2$ and $d_1 = d_2$, the tax yields social optimum for all s.

The corollary follows from the fact that we can impose the optimal monopoly tax on firm 1 if $\Delta < 0$ and $s \geq s_2^{M_1}$, or if $\Delta > 0$ and $0 \leq s \leq s_2^{M_1}$. Accordingly we can impose the optimal monopoly tax on firm 2 if $\Delta > 0$ and $s \geq s_1^{M_1}$. For $\Delta = 0$ and $c_1 \neq c_2$, we know from Proposition 3.1 that only form 1 should produce for all s. Under taxes, however, no firm produces alone if they both produce under *laissez faire*. Only if both firms are alike, we can achieve social optimum under taxes, which is also EBERT's [4] result. We will return to the efficiency issue in section 7. Finally we state:

[12]To see this consider first the natural case where both firms produce for $\tau = 0$. Then firm 1 will drop out first as τ increases. If at all, firm 2 can only drop out whereas firm 2 stays if τ decreases, that is, becomes negative. If $q_2 = 0$ for $\tau = 0$, then it is easy to see that it will produce for no τ.

[13]By shifting this curve to the left and cutting off at $s = 0$ one gets the shape for the other cases. For case a) interchange the subscripts 1 and 2 and shift the curve to the left such that s_1^D and $s_2^{M_1}$ vanish.

Proposition 5.2 *Under Assumption 1 we get $\tau(0) < 0$ for all Δ.*

Thus, for low damage parameters, the firms' pollution will be subsidized. We know that a monopolist or a (Cournot–) oligopoly produce less than the social optimum (which is equal to the competitive output of the lower cost firm). From the theory of regulating monopolies or oligopolies (see BARON and Meyerson [1], or recently EBERT [4]) we know that in the absence of externalities and under complete information, the firms' output is to be subsidized in order to increase welfare. A monopolist can even be brought to produce the competitive output. In our model, the subsidies work indirectly via subsidizing emissions, which stand in fixed proportions to the firms' output.

6 Permits

In this section we assume that the government gives out a number of L pollution permits which may be traded among the firms. Each permit allows a firm to emit one unit of the pollutant. We need not care about whether the permits will be bought from the government and at what price. We could assume that the government distributes them fairly among the firms such that each firm holds $L/2$ permits at the beginning. As we will see, the initial allocation of permits will not effect the outcome. Assume that L be arbitrarily divisible.

6.1 The Firms' Behavior

The process going on in the economy may be divided into 3 steps. At first, the firms hold some initial endowment (l_1, l_2) of permits, with $l_1 + l_2 = L$. In the second step they may trade, that is here, one firm sells some or all permits to the other firm. Firms end up with a new allocation of permits (e_1, e_2) with $e_1 + e_2 = L$. In the third step, firms engage into (Cournot–)competition and choose quantities q_1^N, q_2^N under the constraint

$$q_i^N \leq e_i/d_i \tag{6.1}$$

which is binding if e_i is sufficiently low. To figure out how the firms will trade the permits, denote by $\Pi_i^N(e_1, e_2)$ the profit of firm i if the final allocation of permits in the second step has been (e_1, e_2) and both firms choose Nash–quantities under the constraint (6.1). Observe that there is a gain from trade if and only if there is an allocation (e_1, e_2) such that

$$\Pi_1^N(l_1, l_2) + \Pi_2^N(l_1, l_2) < \Pi_1^N(e_1, e_2) + \Pi_2^N(e_1, e_2)$$

In this case there is T which can be interpreted as *transfer–payment* from firm 1 to firm 2 (which may be negative, of course) such that

$$\begin{aligned} \Pi_1^N(e_1, e_2) + T &> \Pi_1^N(l_1, l_2) \ , \\ \Pi_2^N(e_1, e_2) - T &> \Pi_2^N(l_1, l_2) \ . \end{aligned}$$

How the firms figure out T is nothing we have to care about. For example, they could agree on the Nash–bargaining solution. The maximum gain from trading permits is determined by

$$\max_{e_1,e_2}\left[\Pi_1^N(e_1,e_2)+\Pi_2^N(e_1,e_2)\right] \qquad \text{s.t. } e_1+e_2\le L,\ e_1\ge 0,\ e_2\ge 0. \tag{6.2}$$

Accepting the assumption that firms behave as profit maximizers it is natural to make the following assumption:

Assumption 4 *Firms trade permits in the second phase such that the final allocation (e_1^*, e_2^*) solves (6.2).*

Notice that this assumption allows also for the case that one firm buys all the other firm's permits such that the market ends up with monopoly. And indeed, this will happen for some range of values for L as we will see.

Before the government can solve the problem how to choose the optimal number of permits contingent on s, we have to analyze how the firms will determine the final allocation by solving (6.2). For this consider the following program:

$$\max_{q_1,q_2} P(q_1+q_2)[q_1+q_2]-c_1q_1-c_2q_2 \qquad \text{s.t. } d_1q_1+d_2q_2\le L\ . \tag{6.3}$$

After solving (6.3), we will show that the resulting quantities form a Nash equilibrium, under the constraint that $q_i \le e_i/d_i$. Denote by q_{mon} the monopoly output of the lower cost firm 1 (which is also the monopoly outcome of the horizontally integrated industry). Denote further by $L_{mon} = d_1 q_{mon}$ the number of permits that are at least necessary for producing q_{mon}.

Proposition 6.1 a) *If $\Delta \le 0$, $\forall\, L \ge 0$ the solution of (6.3) is given by* [14]

$$q_1(L) \;=\; \min\left\{q_{mon}\,,\ \frac{L}{d_1}\right\}, \qquad q_2(L)=0$$

b) *If $c_1 < c_2$ and* [15] $\Delta > 0$*, there are $\underline{L}$, $\overline{L}$ with $0 \le \underline{L} < \overline{L}$ such that the solution of (6.3) is given by*

$$\left.\begin{array}{rcl} q_1(L) &=& \min\{q_{mon}\,,\ \frac{L}{d_1}\} \\ q_2(L) &=& 0 \end{array}\right\} \quad \textit{for} \quad L \ge \overline{L}$$

$$\left.\begin{array}{rcl} q_1(L) &>& 0 \\ q_2(L) &>& 0 \end{array}\right\} \quad \textit{for} \quad \overline{L} > L > \underline{L}$$

$$\left.\begin{array}{rcl} q_1(L) &=& 0 \\ q_2(L) &=& \frac{L}{d_2} \end{array}\right\} \quad \textit{for} \quad L \le \underline{L} \qquad \textit{and } d_2 > 0.$$

[14] If both firms are alike, the solution is not unique either firm could buy all the permits.

[15] If $c_1 = c_2$ interchange the names of the firms and apply case a).

Moreover, $q_i(L)$ are continuous in L and $Q(L) := q_1(L)+q_2(L)$ is constant for $\overline{L} \geq L \geq \underline{L}$.

To interpret the proposition: if $\Delta \leq 0$, firm 1 buys all the permits and behaves as a monopolist. If $L > L_{mon}$, firm 1 also buys all the permits but does not use them all. In this case, there is underproduction combined with underpollution. By giving out more permits, however, the government cannot induce the firms to produce more than the monopoly output q_{mon}.

If $\Delta > 0$, the same thing happens as long as $L \geq \overline{L}$. If $\overline{L} \geq L \geq \underline{L}$, the two firms shift production continuously from firm 1 to firm 2 as L decreases, holding total output constant. For $L \leq \underline{L}$, the less polluting firm 2 buys all the permits and produces alone.

Proposition 6.2 *The solution of (6.2) forms a Nash–equilibrium.*

The proof is obvious for $L \geq \overline{L}$ and $L \leq \underline{L}$ since then the firms just produce their monopoly quantities under the constraint that $q_i \leq L/d_i$. The other firm does not hold any permits and hence cannot produce. If $\underline{L} < L < \overline{L}$, for $q_1(L)$ and $q_2(L)$ to form a Nash–equilibrium it is sufficient to show that

$$\frac{\partial \Pi^i}{\partial q_i}(q_1(L), q_2(L)) > 0 \qquad \text{for } i = 1, 2, \tag{6.4}$$

that is, each firm would like to increase quantities, given the other firm produces $q_j(L)$, but cannot since it is constrained by its number of permits. (6.4) will be established in the appendix.

6.2 The Government's program

Given these reactions of the firms when a number of L permits is in the market, and given the damage parameter s, the government has to find the optimal size of L. Further denote $Q(L) := q_1(L) + q_2(L)$, $e_i(L) := d_i q_i(L)$, $i = 1,2$. Hence it has to solve the following program:

$$\max_L W_s^{Per}(L) := \max_L \int_0^{Q(L)} P(z)dz - S(L,s) - c_1 q_1(L) - c_2 q_2(L) \tag{6.5}$$

If we want to emphasize the dependence on the damage parameter s we write $\widetilde{W}^{Per}(L,s)$. Let $L(s)$ denote the optimal number of permits contingent on s, that is, the solution of (6.5). Before we characterize this solution, we state some preparatory notations and lemmata.

Let[16] $\Sigma^D = \{s \mid \exists L^D(s)$ such that $dW_s^{Per}(L^D(s))/dL = 0$ and $q_1(L^D(s)) > 0, q_2(L^D(s)) > 0\}$ be the set of parameters s for which there exists a number of permits which yields a

[16] In the following, the superscript D stands for "Duopoly" again.

<u>local</u> maximum of $W_s^{Per}(L)$ such that both firms produce. Notice that this need not, and in general will not, be a global maximum for all s. Let $\overline{\Sigma}^D$ be the closure of Σ^D. Denote by $L^D(s)$ the solution of $\frac{dW_s^{Per}(L(s))}{dL} = 0$ for all $s \in \overline{\Sigma}^D$, taking right and left derivatives, respectively, on the boundary.

Further let $L^{M_i}(s)$ be the optimal number of permits if only firm i would be around and produce as a monopolist. It is easy to see from the f.o.c.'s that $L^{M_i}(\cdot)$ is decreasing in s, as long as it is binding for the firms, that is, as long as $L^{M_i}(\cdot) \leq L_{mon}$, and also that $L^D(\cdot)$ is decreasing $\forall s \in \overline{\Sigma}^D$. Hence we can define σ_2^D by

$$L^D(\sigma_2^D) = \overline{L} \tag{6.6}$$

and σ_1^D and by

$$L^D(\sigma_1^D) = \underline{L} \text{ if } d_2 > 0 \text{ and } \sigma_1^D = \infty \text{ if } d_2 = 0. \tag{6.7}$$

In words, σ_1^D is the damage parameter where firm 1 just closes if s *increases* towards σ_1^D and $L(s) = L^D(s)$. Similarly, σ_2^D is the damage parameter where firm 2 just closes if s *decreases* towards σ_2^D and $L(s) = L^D(s)$.

Analogously, we define $\sigma_2^{M_1}$ by

$$L^{M_1}(\sigma_2^{M_1}) = \overline{L} \tag{6.8}$$

and $\sigma_1^{M_2}$ and by

$$L^D(\sigma_1^{M_2}) = \underline{L} \text{ if } d_2 > 0 \text{ and } \sigma_1^{M_2} = \infty \text{ if } d_2 = 0. \tag{6.9}$$

In words, $\sigma_1^{M_2}$ is the damage parameter where firm 1 would open up if s fell below $\sigma_1^{M_2}$ and $L(s) = L^{M_2}(s)$. Similarly, $\sigma_2^{M_1}$ is the damage parameter where firm 2 would just open up if s slightly *exceeded* $\sigma_2^{M_1}$ and $L(s) = L^{M_1}(s)$. The next two lemmata are the analoga to Lemma 5.4 and 5.5.

Lemma 6.1 *(With a little abuse of notation) let $W^{M_j}(L,s)$ be the welfare when only firm j is around and reacts as a monopolist upon L, and the damage parameter is s.*

If $\Delta > 0$, then

$$\frac{\partial W^{M_1}}{\partial L}(\overline{L}, s_2^D) > 0 \ , \tag{6.10}$$

and if additionally $d_2 > 0$,

$$\frac{\partial W^{M_2}}{\partial L}(\underline{L}, s_1^D) > 0 \ . \tag{6.11}$$

This implies the following

Lemma 6.2 *If $\Delta > 0$, then*

$$\sigma_2^D < \sigma_2^{M_1} \qquad \text{and if } d_2 > 0 \text{ then} \qquad \sigma_1^D < \sigma_1^{M_2} \ . \tag{6.12}$$

Notice that $\sigma_2^{M_1}$ may be smaller, greater or equal to σ_1^D. Lemma 6.1 says that $L^{M_1}(s)$ is greater than $L^D(s)$ for s close to σ_1^D, and if $d_2 > 0$, then $L^{M_2}(s)$ is greater than $L^D(s)$ for s close to σ_2^D. This implies that like the optimal tax, $L(s)$ must be constant on the interval $[\sigma_1^D, \sigma_1^{M_2}]$. For, if $s = \sigma_1^D$, then $L^D(s) = \underline{L}$, and by Proposition 6.1 b), firm 2 buys all the permits from firm 1. For $s \geq \sigma_1^D$, firm 2 behaves as a monopolist. Forbidding firm 1 to produce, the optimal number of permits equals $L^{M_1}(s)$, which is higher than $\underline{L}$ if s is greater but close to σ_1^D. Giving out $L^{M_2}(s) > \underline{L}$ many permits, however, firm 2 does *not* buy all the permits. Hence, $L(s)$ has to be constant and equal to $\underline{L}$ for $s \in [\sigma_1^D, \sigma_1^{M_2}]$ in order to keep firm 1 out of the market. Notice that this argument is very similar to the optimal linear tax scheme, where the tax rate also has to be constant on certain intervals of damage parameters.

On the other hand, $L(s)$ must be discontinuous somewhere in the interval $(\sigma_2^D, \sigma_2^{M_1})$. To see this, consider first the left hand boundary of this interval, σ_2^D. If we employ the "duopoly–policy" L^D, we get $L^D(\sigma_2^D) = \overline{L}$ and $q_2 = 0$. Employing the monopoly policy L^{M_1} w.r.t. firm 1 we get $L^{M_1}(\sigma_2^D) > \overline{L}$ by Lemma 6.1. Let us assume that $L^{M_1}(\sigma_2^D) < L_{mon}$. Obviously, $L^{M_1}(\cdot)$ is the better policy than $L^D(\cdot)$ for $s = \sigma_2^D$. Hence,

$$\widetilde{W}^{Per}\left(L^{M_1}(\sigma_2^D), \sigma_2^D\right) \quad > \quad \widetilde{W}^{Per}\left(L^D(\sigma_2^D), \sigma_2^D\right) \ .$$

By arguing similarly the other way round, we get for $\sigma_2^{M_1}$:

$$\widetilde{W}^{Per}\left(L^{M_1}(\sigma_2^{M_1}), \sigma_2^{M_1}\right) \quad < \quad \widetilde{W}^{Per}\left(L^D(\sigma_2^{M_1}), \sigma_2^{M_1}\right) \ .$$

Since $L^D(\cdot)$, $L^{M_1}(\cdot)$ and $\widetilde{W}^{Per}(\cdot,\cdot)$ are continuous there must be some intersection $\sigma_{int} \in (\sigma_2^D, \sigma_2^{M_1})$ such that

$$\widetilde{W}^{Per}\left(L^{M_1}(\sigma_{int}), \sigma_{int}\right) \quad = \quad \widetilde{W}^{Per}\left(L^D(\sigma_{int}), \sigma_{int}\right) \ ,$$

and $L(\cdot)$ jumps down from $L^{M_1}(\cdot)$ to $L^D(\cdot)$, at least if $\sigma_2^{M_1} < \sigma_1^D$. In the appendix we will show that this intersection is indeed unique. The case $\sigma_2^{M_1} \geq \sigma_1^D$ is similar and will be treated in the proof of the next proposition which characterizes completely the optimal number of permits as a function of the damage parameter s.

Before doing this, we define σ_{mon} as the damage parameter, from where on the monopolistic firm 1 faces a real capacity constraint if it is regulated by $L^{M_1}(\cdot)$ by

$$L(\sigma_{mon}) \quad := \quad L_{mon}$$

Hence, L^{M_1} is not unique, each $L \geq L_{mon}$ would do the job. For convenience we set

$$L^{M_1}(s) := L_{mon} \qquad s \geq s_{mon} \tag{6.13}$$

Proposition 6.3 *a) If $\Delta \leq 0$, the optimal number of permits as a function of s is given by $L(s) = L^{M_1}(s)\ \forall s \geq 0$. In this case, only firm 1 produces for all $s \geq 0$.*

b) If $\Delta > 0$, the optimal number of permits as a function of s is given by

$$L(s) = \begin{cases} L^{M_1}(s) & \text{for } 0 \leq s \leq \sigma_{int} & \text{(only firm 1 produces)} \\ L^D(s) & \text{for } \sigma_{int} < s \leq \sigma_1^D & \text{(both firms produce, interval may be empty)} \\ \underline{L} & \text{for } \max\left\{\sigma_{int}, \sigma_1^D\right\} < s \leq \sigma_1^{M_2} & \text{(only firm 2 produces)} \\ L^{M_2}(s) & \text{for } s \geq \sigma_1^{M_2} & \text{(only firm 2 produces)} \end{cases} \tag{6.14}$$

where σ_{int} is the solution in s of

$$\widetilde{W}^{Per}\left(L^{M_1}(s), s\right) \stackrel{!}{=} \begin{cases} \widetilde{W}^{Per}\left(L^D(s), s\right) & \text{if } s \leq \sigma_1^D \\ \widetilde{W}^{Per}\left(\underline{L}, s\right) & \text{if } s > \sigma_1^D \end{cases}$$

Proof: a) follows immediately from Proposition 6.1, b) follows also from that result and Lemma 6.2. For $\sigma_2^{M_1} < \sigma_1^D$, the argument has been almost elaborated above. For details, see the appendix.

Observe that apart from the monopoly effect for large values of L, we get the same structure as in social optimum: If $\Delta \leq 0$, the worse firm 2 never produces under the first best as well as under the permit solution. Thus we obtain the following corollary:

Corollary 6.1 *If firm 2 has the worse technology, that is, if $\Delta \leq 0$ the permit solution yields the social optimum for all $s \geq \sigma_{mon}$.*

If $\Delta > 0$, only firm 1 produces for low values of s, both firms produce for intermediate values of s, and only firm 2 produces for high values of s. Output is constant in s when both firms produce. Like in social optimum, production shifts from the lower (production) cost but more polluting firm to the higher cost but less polluting firm. $L(s)$ is depicted in figure 3. However, if $\Delta > 0$, we do not get social optimum under permits for all $s \geq 0$ as the following result shows. Recall that both firms produce in *social optimum* if $s \in (\underline{s}, \overline{s})$ – denote total output on $(\underline{s}, \overline{s})$ by $\tilde{Q}$ –, and under permits if $s \in (\sigma_{int}, \sigma_1^D)$ – denote total output under permits on $(\sigma_{int}, \sigma_1^D)$ by $\tilde{\tilde{Q}}$.

Figure 3 about here.

Proposition 6.4 *i) $\sigma_{int} > \underline{s}$, ii) $\sigma_1^D > \overline{s}$ and iii) $\tilde{Q} > \tilde{\tilde{Q}}$.*

For linear demand and quadratic damage function one can even show that that $\sigma_{int} > 2\underline{s}$, and $\sigma_1^D = 2\overline{s}$, furthermore, $\tilde{Q} = 2\tilde{\tilde{Q}}$.

Corollary 6.2 *If $\Delta > 0$, The permit solution is socially optimal for $s \in [\sigma_{mon}, \underline{s})$ and for $s \geq \sigma_1^{M_1} > \overline{s}$.*

This result seems to be disillusening quite a bit, however, the permit regime is not that bad in comparison with the tax solution. Specially for relatively high values of s it yields better results in terms of welfare than the tax regime does as we will see in the next section.

Notice a final remark on this section. If $s \leq s_{mon}$, we saw that the optimal number of permits is not unique. All $L \geq L_{mon} = d_1 q_{mon}$ lead to the monopoly outcome q_{mon}. Giving out no permits at all, leads to the *laissez faire* Cournot Nash equilibrium. If s is close to 0, therefore, no permits are better than any $L \geq L_{mon}$, whereas for $s \geq s_{mon}$ but close to s_{mon} permits are better. Thus if taxes are not under discussion, but the question is wether permits or not, the optimal permit policy is *laissez faire* up to a certain s_0, and to put up with monopoly for $s_0 \leq s \leq s_{mon}$.[17]

7 Comparison and Discussion of the Policies

Recall for the remainder of the paper that s_1^D, s_2^D, $s_1^{M_2}$, $s_2^{M_1}$ denote the border cases for s if we consider *taxes*. For *permits* we use the Greek σ_1^D, σ_2^D, $\sigma_1^{M_2}$, $\sigma_2^{M_1}$, σ_{int} and σ_{mon}.

Throughout this paper we saw that the sign of Δ played a crucial role in the analysis of the model. If this is not positive, a social planner will not allow firm 2 to produce for any damage parameter s. Under the permit solution, firm 2 also never holds any permits, if we accept Assumption 4. Thus, for $s \geq \sigma_{mon}$, the government can always induce firm 1 to produce the social optimum. For $L \geq L_{mon}$, firm 1 behaves as a monopolist under "laissez faire". The government can not induce the monopolist to produce more by giving out more permits. Thus, for s close to zero the tax regime yields higher welfare than the permit solution. This requires not much of a proof. By giving out more permits than L_{mon} the government can do nothing to increase welfare, whereas it can indirectly subsidize output by negatively taxing, that is, subsidizing pollutants. This seems to be some funny perverse effect of pollution control. But it is simply due to the fact that in absence of negative external effects from production oligopoly produces less than social optimum (cf. [4]). If both firms are alike we even get:

Corollary 7.1 *If both firms are alike, the tax solution yields the socially optimal outcome for all $s \geq 0$. The permit solution is socially optimal only for $s \geq \sigma_{mon}$.*

The permit solution, on the other hand, is better than the tax solution if the social

[17] s_0 is determined by the intersection of welfare under *laissez faire* and welfare under monopoly as a function of s.

damage from pollution is not too small, $s \geq \sigma_{mon}$, and one firm is worse than but not too different from the other firm.

Corollary 7.2 *If firm 2 has the worse technology, that is $\Delta < 0$, but is not too bad such both firms produce (that is $|\Delta|$ is not too large) and $s_2^{M_1} > \sigma_{mon}$, the permit solution is at least as good as the tax solution for $s \geq \sigma_{mon}$ (that is for those s for which the better firm 1 pollutes too much as a monopolist under "laissez faire"), and strictly better for $s \in (\sigma_{mon}, s_2^{M_1})$.*

Let us now turn to the case $\Delta > 0$. Proposition 6.4 showed that under permits firm 2 opens too late and firm 1 closes too late as s increases compared with social optimum. Moreover, the supplied quantity if *both* firms produce ($= \tilde{\tilde{Q}}$) is lower under permits than in social optimum ($= \tilde{Q}$). The next proposition shows that under taxes the situation is even worse in some respect.

Proposition 7.1 *If $\Delta > 0$, then $\sigma_1^{M_2} < s_1^{M_2}$.*

In words, the damage parameter, from where on the socially optimal solution is achieved under permits is smaller than the damage parameter, from where on the optimum is achieved under taxes.

Corollary 7.3 *If $\Delta > 0$, the permit regime achieves the social optimum for a greater range of damage parameters, for which it is desirable that the higher polluting firm shuts down, than the tax regime does.*

In the light of this corollary, the permit solution is not as bad as it seemed to be from Proposition 6.4. From Proposition 7.1 it follows also that the permit solution is better than the tax solution for values slightly *lower* than $\sigma_1^{M_2}$. If s further decreases, welfare under taxes may intersect welfare under permits as the following example demonstrates.

Example 7.1 Let $P(Q) = 1 - Q$, $S(E,s) = \frac{s}{2}E^2$ and $c_1 = 0.25$, $c_2 = 0.5$, $d_1 = 1$, $d_2 = 0.5$. Under this constellation, $\Delta = d_1(1 - c_2) - d_2(1 - c_1) > 0$, and we get $\underline{s} = 2, \overline{s} = 4$, that is, in social optimum both firms are active for $s \in (2,4)$. Under the optimal Pigouvian tax, both firms are active for $s = 0$. Firm 1 closes for $s_1^D = 16$. For $s \in (s_1^D, s_1^{M_2}) = (16, 20)$, the tax is constant and equals $\tau = \tau_1^D = 0.666$. Only when $s \geq 20$, the social optimum is obtained by the Pigouvian tax. From figure 4 we see that there is overproduction for $s \in (0, 2.25)$ and underproduction for $s \in (2.25, 20)$, combined with excess pollution for $s \in (2.25, 6)$ and underpollution for $s \in (0, 0.25) \cup (6, 20)$ (see figure 5). Under permits, social optimum is attained for $s \in [\sigma_{mon}, \sigma_2^D) = (0.125, 2)$ and $s \geq \sigma_1^{M_2} = 12$. For $s \in (2, 12)$ there is underproduction combined with excess pollution for $s \in (\underline{s}, \sigma_{int}) = (2{,}4.2)$ and underpollution for $s \in [\sigma_{int}, \sigma_1^{M_2}) = [4.2, 12)$. For the "most" values of s, welfare is lower under taxes as under permits[18], however, for $s \in (2, 6.5)$, the optimal Pigouvian tax yields a higher welfare than the optimal number of permits (cf. Figure 6). So, no policy is superior in general. Compared with "laissez faire", both solutions yield approximately good results as can be seen from figure 7. Other interesting examples could be provided, however, limits on space force us to close here.

8 A Tax/Subsidy System on pollutants and output yields social optimum – "almost always"

We saw that taxes or permits on pollutants are in general inefficient. Let us assume now that emissions are taxed or subsidized by τ, whereas output is taxed or subsidized by ζ.

Proposition 8.1 *Optimal taxes/subsidies on emission and output yield social optimum if firms engage in Cournot competition and $d_1 \neq d_2$.*

Proof: Since the proof is very easy we present it immediately. Let s be given and let $q_i^*(s)$, $i = 1, 2$ be the socially optimal outputs. Under Cournot competition the f.o.c.'s for the Nash–equilibrium under taxes τ on emissions and ζ on output are:

$$P(q_1 + q_2) + P'(q_1 + q_2)q_1 - c_1 - \zeta - \tau d_1 = 0 \tag{8.1}$$

$$P(q_1 + q_2) + P'(q_1 + q_2)q_2 - c_2 - \zeta - \tau d_2 = 0 \tag{8.2}$$

Setting $q_i = q_i^*(s)$ for $i = 1, 2$, we get two linearly independent equations in τ and ζ if $d_1 \neq d_2$. Since the Nash–equilibrium is unique by Assumption 1 we are done. Q.E.D.

Notice, however, that this result is due to the fact that there are only two firms and exactly two policy tools which can influence the firms costs and hence force the output into the right direction. Hence, the qualifier "almost always" in the headline refers to the exception that $d_1 = d_2$. If additionally $c_1 = c_2$, first best can also be achieved, however, not if $c_1 < c_2$ and $d_1 = d_2$.

9 Final Remarks

We investigated and completely characterized the optimal linear tax on emissions and the optimal number of permits for an asymmetric duopoly. Both regimes do not yield social optimum in general. Especially, the allocation of production turned out to be inefficient

[18] This, of course, does not mean very much since we have no measure on the range of s.

under the optimal tax as well as under permits if both firms are active and if firms are different. The permit regime yields a higher welfare if one firm has a better technology for all s and if the lower cost firm would overpollute as a monopolist. The permit regime is also better than taxes for a greater range of high damage parameters for which the lower cost but worse polluting firm should close down in social optimum. The permit regime is clearly worse if social damage is so low that lower cost firm underproduces (and hence underpollutes) as a monopolist such that pollution should be subsidized under the tax regime. In this case, the lower cost firm exploits the permit regime, by buying all the permits and thereby building up its monopoly position. For intermediate values of s nothing can be said in general! Welfare has to be compared under both regimes. But the optimal size of permits or taxes has to be calculated anyway!

In Section 8 we saw that we can get the efficient outcome by taxing output *and* emissions. We mentioned that this result does not generalize to more than two firms and nonlinear technologies. However, one may improve upon linear taxes by looking for optimal incentive compatible nonlinear taxes. This would in general mean to offer the firms a menu of output *and* emission levels combined with taxes or subsidies. Even so, the investigation of *permits* and *linear taxes*, as done here, is important since those tools are, first, relatively easily enforced and secondly, and more important, become more and more known, better understood, and discussed in the public. Since both tools again are equivalent for sufficiently high damage, and at least keep welfare above zero if optimally calculated, in contrast to *laissez faire*, the answer to those politicians who go on wasting time by struggling about the right tool should be "it does not matter so much what you do, rather do *something*".

A Appendix

Notation: Since we will talk several times oft left sided derivatives (for short: l.s.d.) and right sided derivatives (r.s.d.) we write $f^l(x) := \lim_{h\to 0, h<x} \frac{f(x+h)-f(x)}{h}$ for the l.s.d. and $f^r(x)$, respectively for the r.s.d. .

Proof of Proposition 3.1: If $d_1 \leq d_2$ (and $c_1 < c_2$) or $d_1 < d_2$ and $c_1 = c_2$, it is obvious that only firm 1 should produce for $s > 0$, since it has no higher cost and does not pollute more than firm 2. So let $d_1 > d_2$. F.o.c.s of the Lagrange function w.r.t. q_1 and q_2 yield

$$P(q_1 + q_2) - S_1(E, s) \cdot d_1 - c_1 + \mu_1 = 0 \tag{A.1}$$

$$P(q_1 + q_2) - S_1(E, s) \cdot d_2 - c_2 + \mu_2 = 0 \tag{A.2}$$

Here μ_1, μ_2 are the Kuhn–Tucker multipliers w.r.t. the constraints $q_1 \geq 0$ and $q_2 \geq 0$.

Eliminating $S_1(E, s)$ and assuming $\mu_1 = \mu_2 = 0$ yields

$$0 = (d_1 - d_2)P(q_1 + q_2) - d_1 c_2 + d_2 c_1 < (d_1 - d_2)\overline{p} - d_1 c_2 + d_2 c_1 = \Delta\ . \tag{A.3}$$

Thus, $\Delta > 0$ is necessary for both firms to produce. Furthermore, the first equality in (A.3) implies $P(Q) = (d_1c_2 - d_2c_1)/(d_1 - d_2)$, or that $Q = P^{-1}(d_1c_2 - d_2c_1)/(d_1 - d_2)$ is independent of s. (Notice that $\Delta > 0$ implies $d_1c_2 > d_2c_1$.)

Now assume $\mu_1 \geq 0$, $\mu_2 = 0$, that is, $q_1 = 0$, $q_2 > 0$. Then (A.1) and (A.2) become

$$P(q_2) - S_1(d_2q_2, s) \cdot d_1 - c_1 + \mu_1 = 0 \tag{A.4}$$

$$P(q_2) - S_1(d_2q_2, s) \cdot d_2 - c_2 = 0 \tag{A.5}$$

Eliminating $S_1(E, s)$ yields $\Delta > (d_1 - d_2)P(q_1 + q_2) - d_1c_2 + d_2c_1 = d_2\mu_2 \geq 0$. Thus, $\Delta > 0$ is necessary for firm 2 to produce alone. Hence, for $\Delta \leq 0$ only firm 1 produces in social optimum for any damage function S, and the f.o.c. is $P(q_1) - S_1(d_1q_1, s) \cdot d_1 - c_1 = 0$. From this it follows easily that q_1 decreases as s increases. This proves part a).

Next observe that $q_1 + q_2$ is bounded by $P^{-1}(c_1)$, hence $E = d_1q_1 + d_2q_2$ is bounded. Subtracting (A.2) from (A.1) yields

$$(d_2 - d_1)S_1(d_1q_1 + d_2q_2, s) + c_2 - c_1 + \mu_1 - \mu_2 = 0 \tag{A.6}$$

Since E is bounded, for s sufficiently small, (A.6) cannot have a solution in q_1 and q_2 for $\mu_1 \geq 0$, $\mu_2 = 0$. Hence, $\mu_1 = 0$, $\mu_2 > 0$, implying $q_1 > 0$, $q_2 = 0$.

Since $q_1 + q_2 =: \tilde{Q}$ is constant for $q_1 > 0$, $q_2 > 0$, we have $E > d_2\tilde{Q} > 0$ for $d_2 > 0$. Hence, for large s, (A.6) can only have a solution for $\mu_1 > 0$, $\mu_2 = 0$ implying $q_1 = 0$, $q_2 > 0$. Since S is continuous, there must be $\underline{s}$ with $q_2 = 0$, $\mu_2 = 0$, $\mu_1 = 0$, $q_1 > 0$ and $\overline{s}$ with $q_1 = 0$, $\mu_1 = 0$, $\mu_2 = 0$, $q_2 > 0$ and $q_1 > 0$, $q_2 > 0$ for $s \in (\underline{s}, \overline{s})$. Hence Q is also continuous in s. For $d_2 = 0$, $\overline{s} = \infty$. Finally observe that for $s \in (\underline{s}, \overline{s})$, (A.6) becomes $c_2 - c_1 = (d_1 - d_2)S_1(d_1q_1 + d_2q_2, s)$. Since $c_1 < c_2$ and $\Delta > 0$ imply $d_1 > d_2$, and since Q is constant on $(\underline{s}, \overline{s})$, $q_1(s)$ must be decreasing and $q_2(s)$ must be increasing in s on $(\underline{s}, \overline{s})$. Obviously, also E and W are continuous and decreasing as a function of s when the socially optimal quantities are chosen. Q.E.D.

Proof of Lemma 5.1 We prove it indirectly. Suppose τ' is such that firm 1 just closes as a monopolist, that is, $q_1(\tau') = 0 \ \forall \tau \geq \tau'$, and $q_1(\tau) > 0 \ \forall \tau < \tau' - \varepsilon$, moreover, $q_2(\tau) = 0 \ \forall \tau > \tau' - \varepsilon$ for some $\varepsilon > 0$. Then firm 1's f.o.c. at $(q_1, q_2) = (0, 0)$ is

$$\overline{p} - c_1 - \tau' d_1 = 0 \tag{A.7}$$

Taking the right sided derivative of firm 2's profit function at $(q_1, q_2) = (0, 0)$, we get

$$\overline{p} - c_2 - \tau' d_2 < 0 \tag{A.8}$$

Solving (A.7) for τ' and substituting into (A.8) yields $\Delta < 0$.

Suppose now firm 1 closes first, arguing analogously yields $\Delta > 0$. If both firms close simultaneously, then (A.8) holds with equality. Together with (A.7) we get $\Delta = 0$.

Clearly it cannot happen that firm i just closes at some τ' whereas firm j just opens, otherwise q_j would be increasing in τ for a monopolist, contradicting (5.10). Since $c_1 \leq c_2 < \overline{p}$, firm 1 will produce for $\tau = 0$. Hence it cannot be the case that firm 1 never produces for all τ if $\Delta = 0$. Q.E.D.

Proof of Lemma 5.2 Suppose there is τ_1^D satisfying (5.3). Then the f.o.c's in Nash equilibrium are

$$P(q_2) - c_1 - \tau_1^D d_1 = 0 \tag{A.9}$$

$$P(q_2) + P'(q_2)q_2 - c_2 - \tau_1^D d_2 = 0 \tag{A.10}$$

Eliminating τ_1^D yields $\Delta > P(q_2)[d_1 - d_2] - d_1 c_2 + d_2 c_1 = 0$. Q.E.D.

Proof of Lemma 5.3: (Sketched) Differentiating (5.1) for $i = 1, 2$ w.r.t. τ, adding up both equations and solving for Q' yields

$$Q' = \frac{d_1 + d_2}{3P'(Q) + P''(Q)Q} < 0$$

since $3P'(Q) + P''(Q)Q < 2P'(Q) + P''(Q)Q < 0$ by Assumption 1. To show ii) requires some more effort. Again we differentiate (5.1) for $i = 1, 2$ w.r.t. τ, multiply the first equation by d_1 and the second one by d_2. Then we add up and solve for $d_1 q_1'(\tau) + d_2 q_2'(\tau) = E'(\tau)$. After some manipulations we get

$$E' = \frac{2P'(Q)[d_1^2 + d_2 - d_1 d_2] - d_1 d_2 P''(Q)Q}{P'(Q)[3P'(Q) + P''(Q)Q]} . \tag{A.11}$$

The denominator is positive, the first term of the numerator is negative, but the sign of P'' is undetermined. Here we need the lower bound of Assumption 1. If P'' is sufficiently bounded from below, E' is negative. Q.E.D.

Remark A.1 *Observe, however, the interesting phenomenon that total output of emissions may increase as the tax increases if inverse demand is sufficiently concave! (Similar results have been found by* EBERT *[3] and* ENDRES *[5]).*

Proof of Lemma 5.4 We show c). The remaining claims are demonstrated analogously. Since $\tau(s)$ solves (5.2), we have $\frac{dW^{PT}}{d\tau}(\tau^D(s)) = 0$, if $q_i(\tau^D(s)) > 0$, $i = 1, 2$. For $s = s_1^D$ the left sided derivative of W^{PT} equals zero:

$$\frac{dW^{PT}}{d\tau}(\tau_1^D) = 0 \tag{A.12}$$

Since also $q_1(\tau_1^D) = 0$, (A.12) becomes (writing just τ instead of τ_1^D to save space):

$$P(q_2(\tau))[q_1^l(\tau) + q_2^l(\tau)] - S_1(d_2 q_2(\tau), s)[d_1 q_1^l(\tau) + d_2 q_2^l(\tau)] - c_1 q_1^l(\tau) - c_2 q_2^l(\tau) = 0$$

or

$$S_1(d_2q_2^N(\tau),s) = \frac{P(q_2^N(\tau))[q_1^l(\tau)+q_2^l(\tau)] - c_1q_1^l(\tau) - c_2q_2^l(\tau)}{d_1q_1^l(\tau)+d_2q_2^l(\tau)} \tag{A.13}$$

Consider now the welfare function $W_s^{M_2}$ when only firm 2 produces and is taxed as a monopolist. First order condition for the optimal monopoly tax yields

$$\frac{dW_s^{M_2}}{d\tau}(\tau) = [P(q_2(\tau)) - S_1(d_2q_2(\tau),s)d_2 - c_2]q_2^\tau(\tau) = 0 \tag{A.14}$$

where q_2^τ denotes the r.s.d. . Substituting (A.13) into (A.14) we get

$$\frac{dW_s^{M_2}}{d\tau}(\tau) = \frac{q_1^l(\tau_1^D)\cdot q_2^\tau(\tau_1^D)}{d_1q_1^l(\tau_1^D)+d_2q_2^l(\tau_1^D)}[P(q_2(\tau_1^D))[d_1-d_2] - (d_1c_2 - d_2c_1)] \tag{A.15}$$

On the other hand, we have for τ_1^D:

$$P(q_2) - c_1 - \tau_1^D d_1 = 0 \tag{A.16}$$

$$P(q_2) + P'(q_2)q_2 - c_2 - \tau_1^D d_2 = 0 \tag{A.17}$$

Eliminating τ_1^D yields $[d_1-d_2]P(q_2) - d_1c_2 + d_2c_1 + d_1P'(q_2)q_2 = 0$, hence, $[d_1-d_2]P(q_2(\tau_1^D)) - d_1c_2 + d_2c_1 > 0$. Now, $q_2^\tau(\tau_1^D) < 0$ by (5.9), $q_1^l(\tau_1^D) < 0$ since firm 1 closes down at $\tau = \tau_1^D$. The denominator equals $E'(\tau_1^D)$ which is also negative by Lemma 5.3), ii). Hence, the whole derivative is negative. Q.E.D.

Proof of Lemma 5.5 We show that $\Delta > 0$ implies $s_1^D < s_1^{M_2}$. The remaining claims are demonstrated analogously. Since $\frac{dW^{PT}}{d\tau}(\tau_1^D) < 0$ if firm 2 is regulated as a monopolist (suppose firm 1 is not existent for a moment) the *optimal* tax to regulate a monopoly is lower than τ_1^D by the last lemma. Since $\tau^M(s)$ is increasing in s by (5.11), $s_1^{M_2}$ must be greater than s_1^D. Q.E.D.

Proof of Proposition 5.2 For $s = 0$ the f.o.c. of the government's program leads to

$$P(Q)Q'(\tau) - c_1q_1'(\tau) - c_2q_2'(\tau) = 0 \tag{A.18}$$

Adding up the Nash–equilibrium conditions for both firms yields

$$2P(Q) + P'(Q)(q_1+q_2) - c_1 - c_2 - \tau(d_1+d_2) = 0 \tag{A.19}$$

Substituting (A.18) into (A.19) and manipulating yields

$$\frac{(c_1-c_2)(q_1'(\tau)-q_2'(\tau))}{Q'(\tau)} + P'(Q)Q = \tau(d_1+d_2) \tag{A.20}$$

Since $c_1 < c_2$, $Q' < 0$ and $P' < 0$, the L.H.S. of (A.20) is negative if $q_1' < q_2'$. To get this, differentiate the Nash–equilibrium conditions of both firms w.r.t. τ and subtract one of the other. This yields after rearranging:

$$q_1' - q_2' = \frac{d_1-d_2}{P'(Q)} - \frac{P''(Q)(q_1-q_2)}{P'(Q)} \tag{A.21}$$

It is easy to verify that $c_1 < c_2$ implies $q_1 > q_2$ in Nash equilibrium. Then the R.H.S. of (A.21) is neagtive, if P is not too concave.

Proof of Proposition 6.1 If $d_1 < d_2$, clearly $q_2 = 0\ \forall L \geq 0$, hence let $d_1 > d_2$. The f.o.c.s of the program (6.3) are

$$P'(Q) \cdot Q + P(Q) - c_1 - \lambda d_1 + \mu_1 = 0 \tag{A.22}$$

$$P'(Q) \cdot Q + P(Q) - c_2 - \lambda d_2 + \mu_2 = 0 \tag{A.23}$$

where λ is the Kuhn–Tucker multiplier w.r.t. $d_1q_1 + d_2q_2 \leq L$, and μ_1, μ_2 are the multipliers w.r.t. $q_1 \geq 0$, $q_2 \geq 0$. Suppose now $\lambda \neq 0$, $\mu_1 \geq 0$, and $\mu_2 = 0$, hence $q_2 > 0$. Eliminating λ from (A.22) and (A.23) yields:

$$(d_1 - d_2)P(Q) - d_1c_2 + d_2c_1 + (d_1 - d_2)P'(Q) - d_2\mu_2 = 0\ .$$

The L.H.S. is smaller than Δ, hence firm 2 will never produce anything if $\Delta \leq 0$. If $\Delta \leq 0$, and $L > L_{mon}$, clearly firm 1 does not use all the permits. If $L \leq L_{mon}$, firm 1's output is constrained by L/d_1. Suppose now $\Delta > 0$. If $0 \leq \lambda < (c_2 - c_1)/(d_1 - d_2)$, clearly there is no solution for $\mu_1 = \mu_2 = 0$. It easy to see that then $q_2 = 0$ and $q_1 = \min\{q_{mon}\ ,\ L/d_1\}$. If $\lambda = (c_2 - c_1)/(d_1 - d_2)$, Q is independent of L. This follows by subtracting (A.22) from (A.23), but q_1 decreases, q_2 increases if L decreases. Obviously there is $\underline{L}$ such that $q_2 = 0$, $q_1 = L/d_1$ and $\overline{L}$ such that $q_1 = 0$, $q_2 = L/d_2$, and $q_1 > 0$, $q_2 > 0$ for all $\overline{L} > L > \underline{L}$. For $\lambda > (c_2 - c_1)/(d_1 - d_2)$, clearly q_1 must be zero and $q_2 = L/d_2$. Q.E.D.

Proof of Proposition 6.2 Let $\overline{q}_1 := q_1(L)$, $\overline{q}_2 := q_2(L)$ be the solution of (6.3) and let $\overline{Q} := \overline{q}_1 + \overline{q}_2$. Then

$$\frac{\partial \Pi_i(\overline{q}_1, \overline{q}_2)}{\partial q_i} = P'(\overline{Q})\overline{q_i} + P(\overline{Q}) - c_i \geq P'(\overline{Q})\overline{Q} + P(\overline{Q}) - c_i - \lambda d_i - \mu_i$$

(and strictly greater if $q_j > 0$). The R.H.S. is a f.o.c. of the program (6.3) and equals zero. Hence firm i is either a monopolist or it would like to increase output given the output $\overline{q}_j > 0$ of firm j. Q.E.D.

Proof of Lemma 6.1 We show (6.11). (6.10) is demonstrated in the same way by interchanging the indices 1 and 2. The proof works similar to the proof of Lemma 5.5. We will show that

$$\frac{\partial \widetilde{W}^{Per}}{\partial L}(L^D(\sigma_1^D), \sigma_1^D) > 0\ ,$$

if firm 1 could be forbidden to produce.

By definition of $L^D(s)$ we have for the r.s.d. at $s = \sigma_1^D$: $\frac{\partial \widetilde{W}^{Per}}{\partial L}(\underline{L}, \sigma_1^D) = 0$. Since $q_1(\underline{L}) = 0$, we get $0 = \frac{\partial \widetilde{W}^{Per}}{\partial}(\underline{L}, \sigma_1^D) =$

$$P(q_2(\underline{L}))[q_1^r(\underline{L}) + q_2^r(\underline{L})] - S_1(\underline{L}, \sigma_1^D)[d_1q_1^r(\underline{L}) + d_2q_2^r(\underline{L})] - c_1q_1^r(\underline{L}) - c_2q_2^r(\underline{L}) \tag{A.24}$$

Since $Q(L)$ is constant on $[\underline{L}, \overline{L}]$, we get $q_1'(\underline{L}) + q_2'(\underline{L}) = Q'(L) = 0$ on $[\underline{L}, \overline{L}]$, taking the r.s.d. at $\underline{L}$. Hence $q_1^r(\underline{L}) = -q_2^r(\underline{L})$. Moreover, $d_1 q_1'(\underline{L}) + d_2 q_2'(\underline{L}) = E'(L) = 1$ on $[\underline{L}, \overline{L}]$, taking the r.s.d. at $\underline{L}$. Together this yields $q_1^r(\underline{L}) = 1/(d_1 - d_2)$ and $q_2^r(\underline{L}) = -1/(d_1 - d_2)$. Hence (A.24) reduces to

$$S_1(\underline{L}, \sigma_1^D) = \frac{c_2 - c_1}{d_1 - d_2} \tag{A.25}$$

On the other hand, forbidding firm 1 to produce and calculating the l.s.d. of $\widetilde{W}^{Per}$ w.r.t. L at $(\underline{L}, \sigma_1^D)$ we get

$$\frac{\partial \widetilde{W}^{Per}}{\partial L}(\underline{L}, \sigma_1^D) = [P(q_2(\underline{L})) - S_1(\underline{L}, \sigma_1^D) d_2 - c_2] q_2^l(\underline{L})$$

Plugging in (A.25) yields

$$\begin{aligned} &= \left[P(q_2(\underline{L})) - d_2 \frac{c_2 - c_1}{d_1 - d_2} - c_2\right] q_2^l(\underline{L}) \\ &= \frac{1}{d_1 - d_2} \left[(d_1 - d_2) P(q_2(\underline{L})) - d_1 c_2 + d_2 c_1\right] q_2^l(\underline{L}) \end{aligned}$$

Since $d_1 - d_2 > 0$ and $q_2^l(\underline{L}) > 0$, it remains to show that the term in brackets is positive. But the f.o.c.s of the program (6.3) for $L = \underline{L}$ are

$$\begin{aligned} P(q_2) - c_1 - \lambda d_1 &= 0 \\ P(q_2) + P'(q_2) q_2 - c_2 - \lambda d_2 &= 0 \end{aligned}$$

Eliminating λ yields $(d_1 - d_2) P(q_2(\underline{L})) - d_1 c_2 + d_2 c_1 + P'(q_2(\underline{L})) q_2(\underline{L}) = 0$. Since $P' < 0$, the L.H.S. is smaller than $(d_1 - d_2) P(q_2(\underline{L})) - d_1 c_2 + d_2 c_1$. Q.E.D.

Proof of Lemma 6.2 (6.11) implies $L^{M_2}(\sigma_1^D) > L^D(\sigma_1^D) = \underline{L}$ and hence $\sigma_1^D < \sigma_1^{M_2}$ since $L^{M_2}(\cdot)$ is decreasing (if it is binding for firm 2). (6.10) implies $L^{M_2}(\sigma_1^D) > L^D(\sigma_1^D) = \underline{L}$ and hence $\sigma_2^D > \sigma_2^{M_1}$ since $L^{M_1}(\cdot)$ is decreasing (if it is binding for firm 1). Q.E.D.

Proof of Proposition 6.3 **case a):** $\sigma_{mon} \leq \sigma_2^D$, $\sigma_2^{M_1} < \sigma_1^D$. This case has almost been proven in the text. For $s \leq \sigma_{mon}$, $L(s) = L^{M_1}(s) = L_{mon}$. For $s > \sigma_{mon}$, L^{M_1}, L^{M_2} and L^D are continuous and strictly decreasing. Further, $q_2(\overline{L}) = 0$, and $q_2(L) > 0$ for $\overline{L} > L > \underline{L}$. Now, $\overline{L} = L^D(\sigma_2^D) = L^{M_1}(\sigma_2^{M_1})$. By Lemma 6.1, we get

$$\widetilde{W}(L^{M_1}(s), s) > \widetilde{W}(L^D(s), s)$$

for $\sigma_2^D \leq s < \sigma_2^D + \varepsilon$, if $\varepsilon > 0$ and not too large. Hence $L(s) = L^{M_1}(s)$ for $\sigma_2^D \leq s < \sigma_2^D + \varepsilon$. On the other hand, $q_2\left(L^{M_1}(s)\right) > 0$ if $s > \sigma_2^{M_1}$. Since $\sigma_2^{M_1} < \sigma_1^D$, also $q_1\left(L^{M_1}(s)\right) > 0$ if $s > \sigma_2^{M_1}$. But if both firms produce, $L^D(s)$ is optimal by definition. Hence

$$\widetilde{W}(L^{M_1}(s), s) < \widetilde{W}(L^D(s), s)$$

for $\sigma_2^{M_1} - \varepsilon < s \leq \sigma_2^{M_1}$, if $\varepsilon > 0$ and not too large. Since $\widetilde{W}(L^{M_1}(s), s)$ and $\widetilde{W}(L^D(s), s)$ are continuous, there must be a σ_{int} such that

$$\widetilde{W}(L^{M_1}(\sigma_{int}), \sigma_{int}) = \widetilde{W}(L^D(\sigma_{int}), \sigma_{int}) \ .$$

If we can show that s_{int} is unique, we clearly get $L(s) = L^{M_1}(s)$ for $s \leq \sigma_{int}$ and $L(s) = L^D(s)$ for $\sigma_{int} < s \leq \sigma_1^D$. To establish uniqueness of σ_{int}, it suffices to show that the slope of $\widetilde{W}(L^{M_1}(s), s)$ is steeper on $[s_2^D, s_2^{M_1}]$ then the slope of $\widetilde{W}(L^D(s), s)$. By the envelope theorem we get

$$\begin{aligned} \frac{d\widetilde{W}}{ds}(L^{M_1}(s), s) &= -S_2(L^{M_1}(s), s) \\ \frac{d\widetilde{W}}{ds}(L^D(s), s) &= -S_2(L^D(s), s) \end{aligned}$$

Since $S_{21}(L, s) = S_{12}(L, s) > 0$ for $L, s > 0$, we are done if we can show that $L^{M_1}(s) > L^D(s)$ on the relevant domain. The f.o.c.s for $L^{M_1}(s)$ and $L^D(s)$ imply that

$$S_1(L^{M_1}(s), s) = \frac{P\left(\frac{L^{M_1}(s)}{d_1}\right) - c_1}{d_1} \tag{A.26}$$

$$S_1(L^D(s), s) = \frac{c_2 - c_1}{d_1 - d_2} \tag{A.27}$$

Since $S_{11}(L, s) > 0$ for $L, s > 0$, $L^{M_1}(s) > L^D(s)$ holds if the R.H.S. of (A.26) is greater than the R.H.S. of (A.27). But we know already by Lemma 6.1 that this holds for $s = s_2^D$. The final step is to show that the R.H.S. of (A.26) increases in s. Since $P' < 0$ we have to show that L^{M_1} decreases. Differentiating (A.26) w.r.t. s yields

$$\frac{dL^{M_1}}{ds} = \frac{S_{12} D_1^2}{P' - S_{11} d_1^2} < 0 \ .$$

This establishes the behavior of $L(s)$ for $s \leq \sigma_1^D$.

For $s = \sigma_1^D$, we have $L^D(\sigma_1^D) = \underline{L}$, hence $q_1(\underline{L}) = 0$. For $L < \underline{L}$ firm 2 is a monopolist. In the absence of firm 1, we had $L(s) = L^{M_2}(s)$. By Lemma 6.2, however, and since $L^{M_2}(s)$ is decreasing, we get $L^{M_2}(s) > \underline{L}$ for $\sigma_1^D \leq s < \sigma_1^D + \varepsilon$ for appropriate ε. Hence firm 2 would operate if $L(s) = L^{M_2}(s)$ and $\sigma_1^D \leq s < \sigma_1^D + \varepsilon$. But then, welfare could be increased by decreasing L. Hence $L(s) = \underline{L}$ for $\sigma_1^D \leq s \leq \sigma_1^{M_2}$. For $s > \sigma_1^{M_2}$, we have $L^{M_2}(s) > \underline{L}$ by definition of $\sigma_1^{M_2}$. Hence, $L(s) = L^{M_2}(s)$ for $s > \sigma_1^{M_2}$.

case b): $\sigma_{mon} > \sigma_2^D$, $\sigma_2^{M_1} \geq \sigma_1^D$. If $\widetilde{W}(L_{mon}, \sigma_{mon}) > \widetilde{W}(L^D(\sigma_{mon}), \sigma_{mon})$ we are done, since then $\sigma_{int} > \sigma_{mon}$. If $\widetilde{W}(L_{mon}, \sigma_{mon}) \leq \widetilde{W}(L^D(\sigma_{mon}), \sigma_{mon})$, it again suffices to show that the slope of $\widetilde{W}(L_{mon}, s)$ is steeper than the slope of $\widetilde{W}(L^D(s), s)$ for $s \in (s_2^D, \sigma_{mon})$. But $d\widetilde{W}(L_{mon}, s)/ds = -S_2(L_{mon}, s)$. Since $L_{mon} > L^D(s)$ we are done by the same arguments in case a).

case c): $\sigma_{mon} \leq \sigma_2^D$, $\sigma_2^{M_1} \geq \sigma_1^D$. In order to establish the unique existence of $\sigma_{int} \in (\sigma_2^D, \sigma_2^{M_1})$ such that $\widetilde{W}(L^{M_1}(\sigma_{int}), \sigma_{int}) = \widetilde{W}(L^D(\sigma_{int}), \sigma_{int})$ or $\widetilde{W}(L^{M_1}(\sigma_{int}), \sigma_{int}) = \widetilde{W}(\underline{L}, \sigma_{int})$, we first have to show that the l.s.d. $\partial \widetilde{W}^{Per}(\underline{L}, \sigma_2^{M_1})/\partial L$ is negative. Now,

$$\begin{aligned}
&\frac{\partial \widetilde{W}(\overline{L}, \sigma_2^{M_1})}{\partial L} \\
= \ & P(q_1(\overline{L}))[q_1^l(\overline{L}) + q_l^r(\overline{L})] - S_1(\overline{L}, \sigma_2^{M_1})[d_1 q_1^l(\overline{L}) + d_2 q_2^l(\overline{L})] - c_1 q_1^l(\overline{L}) - c_2 q_2^l(\overline{L}) \\
= \ & -S_1(\overline{L}, \sigma_2^{M_1}) + \frac{c_2 - c_1}{d_1 - d_2}
\end{aligned} \tag{A.28}$$

since $Q'(L) = 0$, $E'(L) = 1$ and $q_1^r(L) = 1/(d_1 - d_2)$, $q_2^r(L) = -1/(d_1 - d_2)$ on $[\underline{L}, \overline{L}]$.

On the other hand, taking the r.s.d. with respect to L at $(\overline{L}, \sigma_2^{M_1})$ we get

$$\begin{aligned}
[P(q_1(\overline{L})) - S_1(\overline{L}, \sigma_2^{M_1})d_1 - c_1]q_1^r(\overline{L}) &= 0 \\
\Leftrightarrow \quad S_1(\overline{L}, \sigma_2^{M_1}) &= \frac{P(q_1(\overline{L})) - c_1}{d_1}
\end{aligned} \tag{A.29}$$

Substituting (A.29) into (A.28) yields

$$\begin{aligned}
\frac{\partial \widetilde{W}^{Per}(\overline{L}, \sigma_2^{M_1})}{\partial L} &= -\frac{P(q_1(\overline{L})) - c_1}{d_1} + \frac{c_2 - c_1}{d_1 - d_2} \\
&< -\frac{P(q_1(\overline{L})) - c_1}{d_1} + \frac{c_2 - c_1}{d_1 - d_2} - \frac{P'(q_1(\overline{L}))q_1(\overline{L})}{d_1 - d_2} = 0
\end{aligned}$$

where the last equality again follows from the f.o.c.s of the program (6.3) for $L = \overline{L}$. This establishes existence.

To show uniqueness, we are done if $\widetilde{W}(L^{M_1}(s_2^{M_1}), s_2^{M_1}) \geq \widetilde{W}(L^D(s_2^{M_1}), s_2^{M_1})$ since then $\sigma_{int} \leq \sigma_2^{M_1}$. Suppose now the contrary. Again it suffices to show that the slope of $\widetilde{W}(L^{M_1}(s), s)$ is steeper than the slope of $\widetilde{W}(\underline{L}, s)$ on the interval $(s_2^{M_1}, s_2^D)$. But $d\widetilde{W}(\underline{L}, s)/ds = -S_2(\underline{L}, s)$. Since $L^{M_1}(s) > \underline{L}$ for $s < s_2^{M_1}$ and arguing as in case a) we are done.

The remaining arguments also work as in case a).

case d): $\sigma_{mon} > \sigma_2^D$, $\sigma_2^{M_1} < \sigma_1^D$. Combine the arguments from the previous cases. Q.E.D.

Proof of Proposition 6.4 Since $\sigma_{int} > \sigma_2^D$, we show that $\sigma_2^D > \underline{s}$. The proof for $\sigma_1^D > \overline{s}$ works the same. Recall that Q is constant on $[\underline{s}, \overline{s}]$ in social optimum, call it $\tilde{Q}$. Recall also that Q is constant on $[\underline{L}, \overline{L}]$ for the solution of (6.3), call it $\tilde{\tilde{Q}}$. In social optimum the f.o.c.s at $s = \underline{s}$ (taking r.s.d.'s) yield

$$P(\tilde{Q}) - S_1(d_1\tilde{Q}, \underline{s})d_1 - c_1 = 0 \tag{A.30}$$

$$P(\tilde{Q}) - S_1(d_1\tilde{Q}, \underline{s})d_2 - c_2 = 0 \tag{A.31}$$

Eliminating $P(\tilde{Q})$ yields

$$S_1(d_1\tilde{Q},\underline{s}) = \frac{c_2 - c_1}{d_1 - d_2} \tag{A.32}$$

The f.o.c. of the government's program for permits at $s = \sigma_1^D$ yields

$$S_1(d_1\tilde{\tilde{Q}}, s_1^D) = \frac{c_2 - c_1}{d_1 - d_2} \tag{A.33}$$

Since $S_{12} > 0$, it remains to show $\tilde{Q} > \tilde{\tilde{Q}}$. Eliminating $S_1(d_1\tilde{Q},\underline{s})$ from (A.30) and (A.31) yields $P(\tilde{Q}) = \frac{d_1c_2 - d_2c_1}{d_1 - d_2}$. Eliminating the Lagrange multiplier (which is the shadow damage) in program (6.3) we get

$$P(\tilde{\tilde{Q}}) = \frac{d_1c_2 - d_2c_1}{d_1 - d_2} - P'(\tilde{\tilde{Q}})\tilde{\tilde{Q}} > \frac{d_1c_2 - d_2c_1}{d_1 - d_2} = P(\tilde{Q})\,. \tag{A.34}$$

Q.E.D.

Proof of Proposition 7.1 We know from Proposition 6.3 that the aggregate output $\tilde{\tilde{Q}}$ under permits on the interval $[\sigma_1^D, \sigma_1^{M_2}]$ equals $q_2(\underline{L})$ and from the proof of Proposition 6.4 we know that $\tilde{\tilde{Q}}$ is determined by the first equation in (A.34). On the other hand, if the tax is τ_1^D, such that firm 1 just closes, the Nash equilibrium conditions yield

$$\begin{aligned} P(q_2(\tau_1^D)) + P'(q_2(\tau_1^D))q_2(\tau_1^D) - c_2 - \tau_1^D d_2 &= 0 \\ P(q_2(\tau_1^D)) - c_1 - \tau_1^D d_1 &= 0 \end{aligned}$$

Eliminating τ_1^D yields

$$P(q_2(\tau_1^D)) = \frac{d_1c_2 - d_2c_1}{d_1 - d_2} - \frac{d_1}{d_1 - d_2}P'(q_2(\tau_1^D))q_2(\tau_1^D) \tag{A.35}$$

Since $\frac{d_1}{d_1 - d_2} > 1$ for $d_2 > 0$, we get $q_2(\tau_1^D) < \tilde{\tilde{Q}} = q_2(\underline{L})$ by virtue of (A.34). Now, for $s = s_1^{M_2}$, and $s = \sigma_1^{M_2}$, respectively, the f.o.c. for the government w.r.t. taxes, and permits, respectively, are those of social optimum:

$$P(q_2(\tau_1^D)) - S_1(d_1\tilde{\tilde{Q}}, s_1^{M_2}) - c_2 = 0 \tag{A.36}$$

$$P(\tilde{\tilde{Q}}) - S_1(d_1\tilde{\tilde{Q}}, \sigma_1^{M_2}) - c_2 = 0 \tag{A.37}$$

Since $Q(s)$ is decreasing in s for social optimum if only firm 2 produces and since $q_2(\tau_1^D) < \tilde{\tilde{Q}}$, we get $s_1^{M_2} > \sigma_1^{M_2}$. Q.E.D.

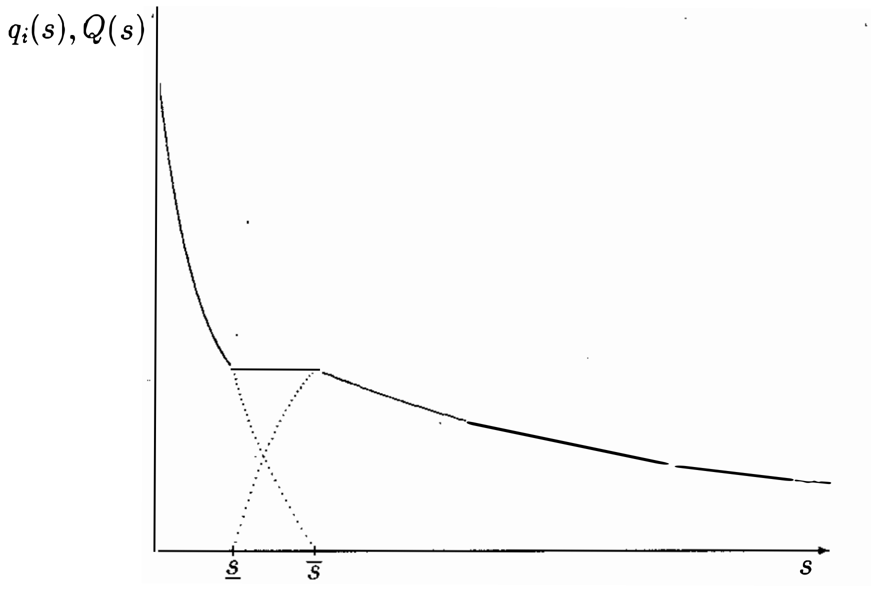

Figure 1: *The quantities in social optimum as a function of s if* $d_1(\bar{\bar{P}}-c_2)-d_2(\bar{\bar{P}}-c_1) > 0$. *The solid line depicts aggregate output which equals* $q_1(s)$ *for* $s \leq \underline{s}$ *and* $q_2(s)$ *for* $s \geq \overline{s}$. *The dotted lines depict* q_1 *and* q_2 *for* $\underline{s} < s < \overline{s}$.

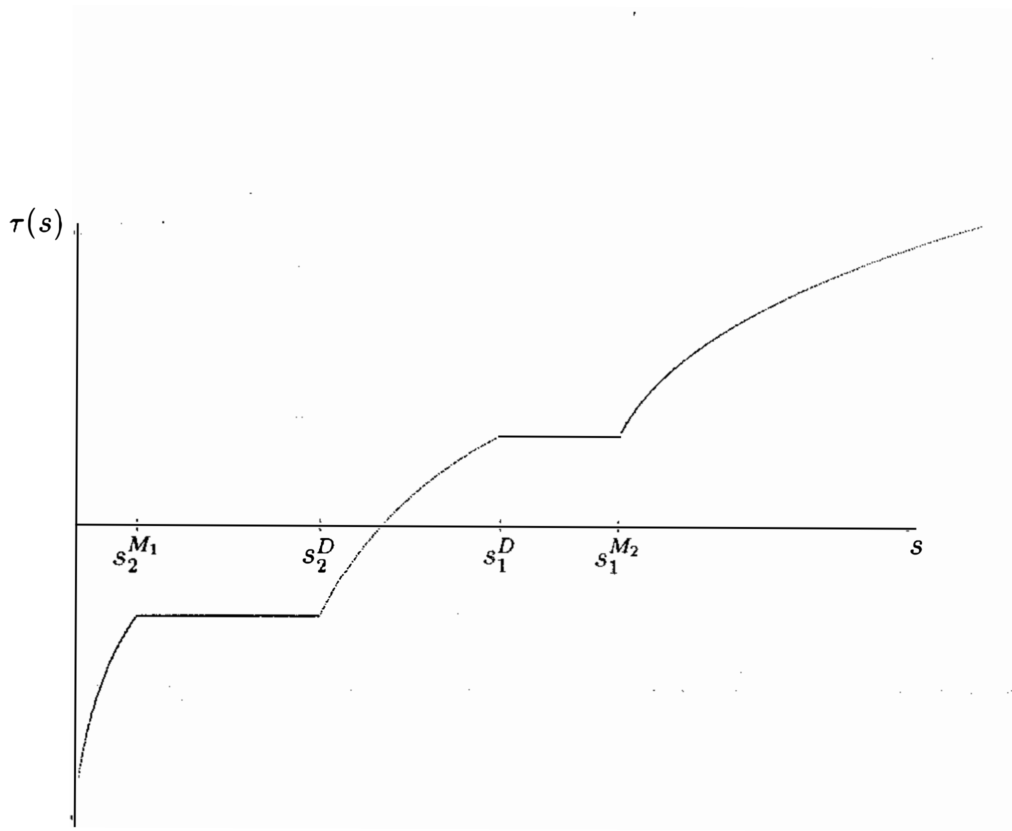

Figure 2: *The optimal linear taxrate as a function of s.*

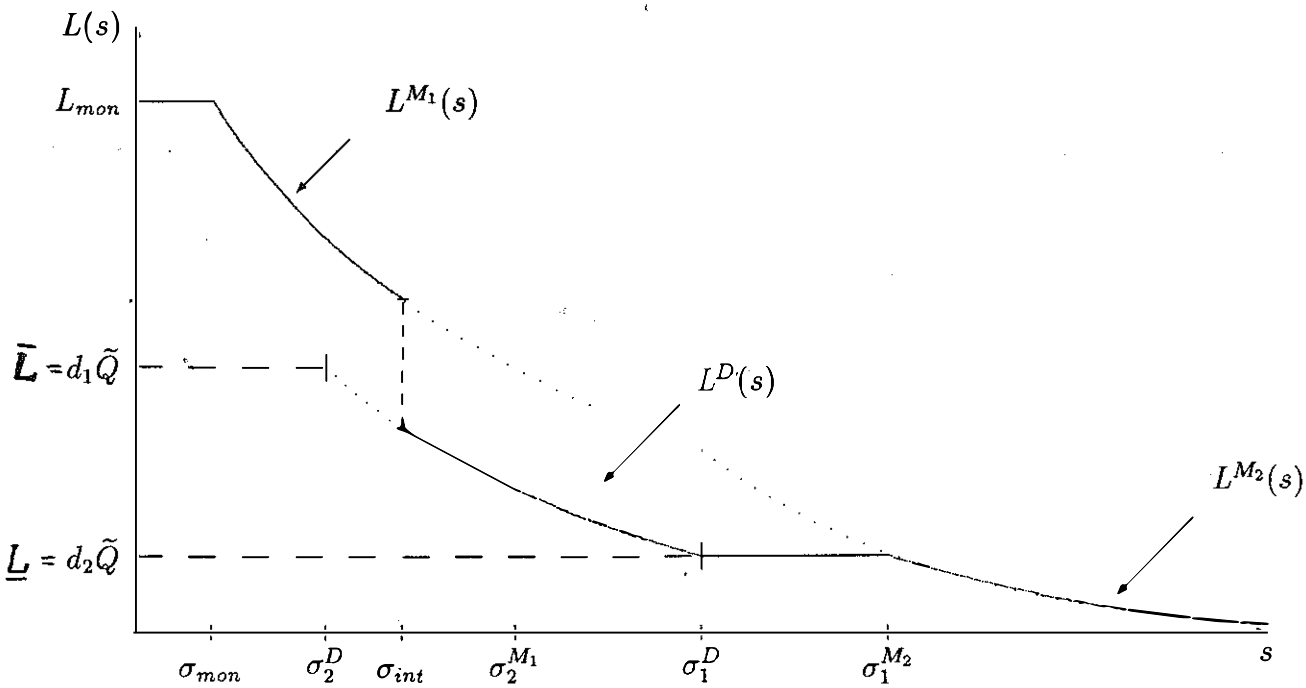

Figure 3: *The solid line depicts the optimal number of permits as a function of s if* $\Delta > 0$.

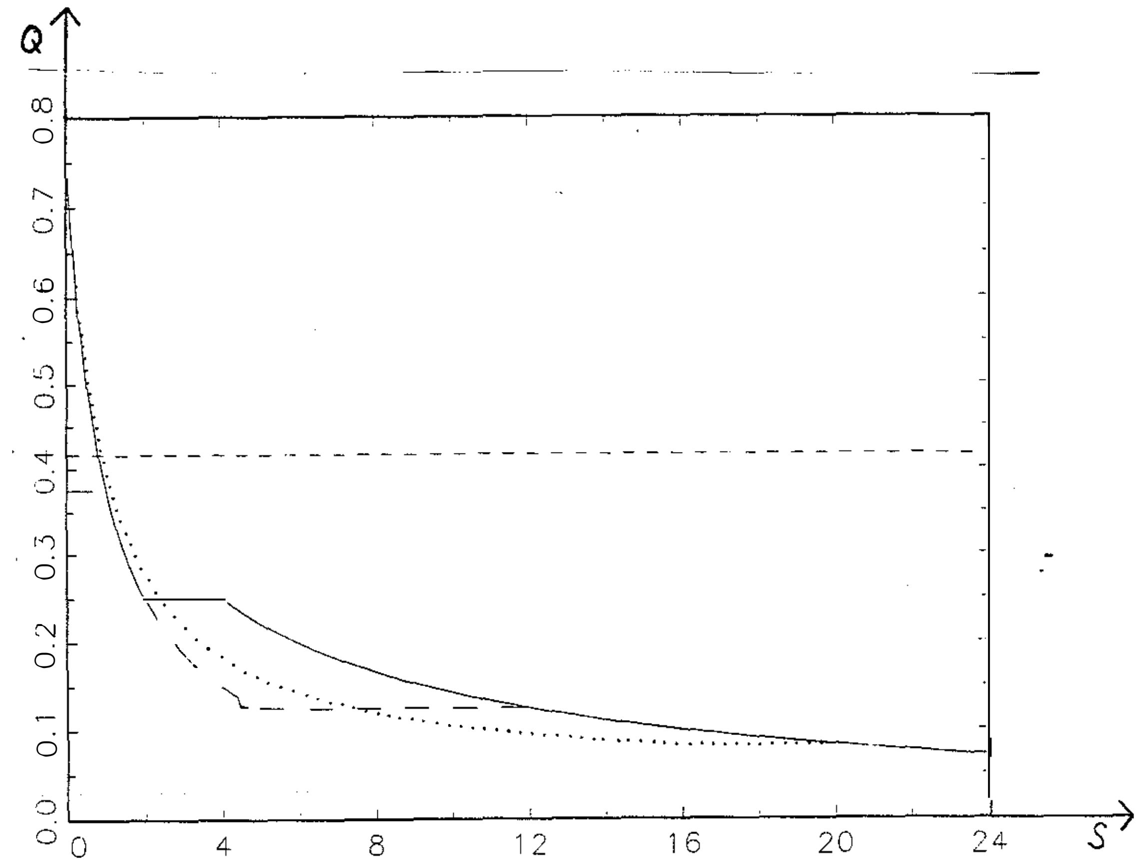

Figure 4: *Aggregate quantities for the marketable output commodity. The solid line depicts the social optimum, the "big dashed" line is for the permit solution, the dotted line for the tax solution, the "small dashed" line denotes "laissez faire".*

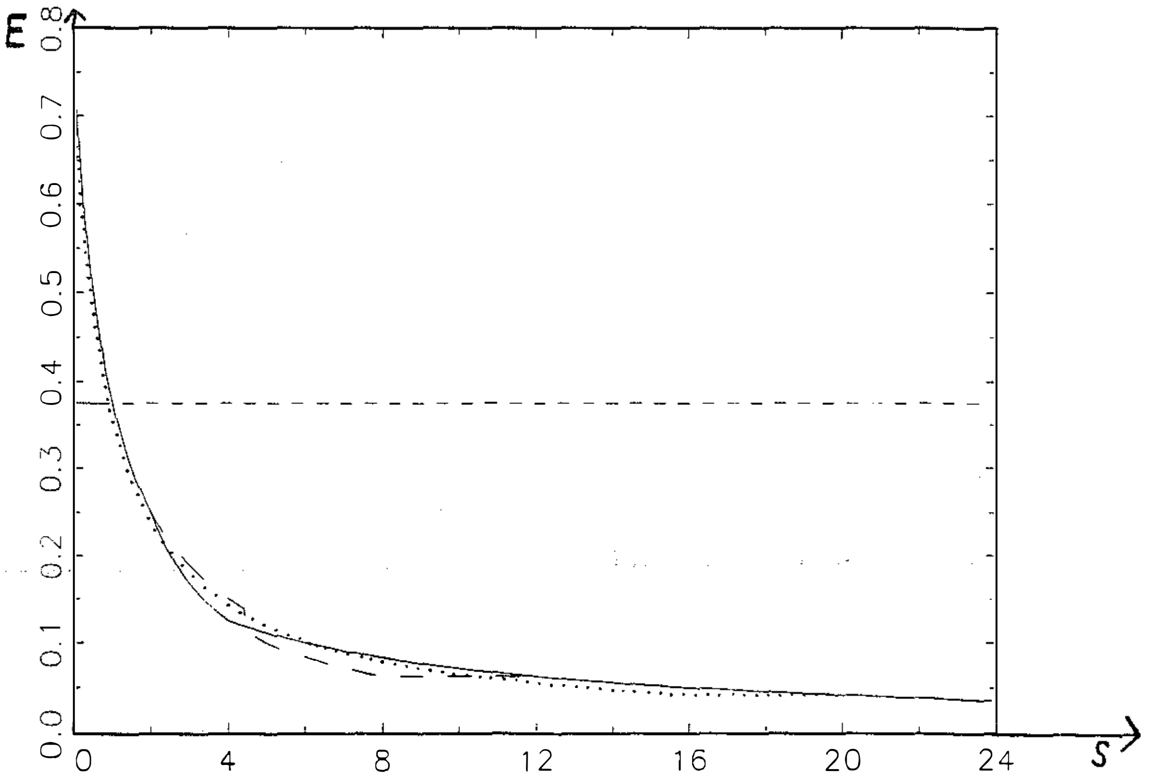

Figure 5: *Aggregate emissions, solid line: social optimum, "big dashed" line: permits, dotted line: taxes, "small dashed" line: "laissez faire".*

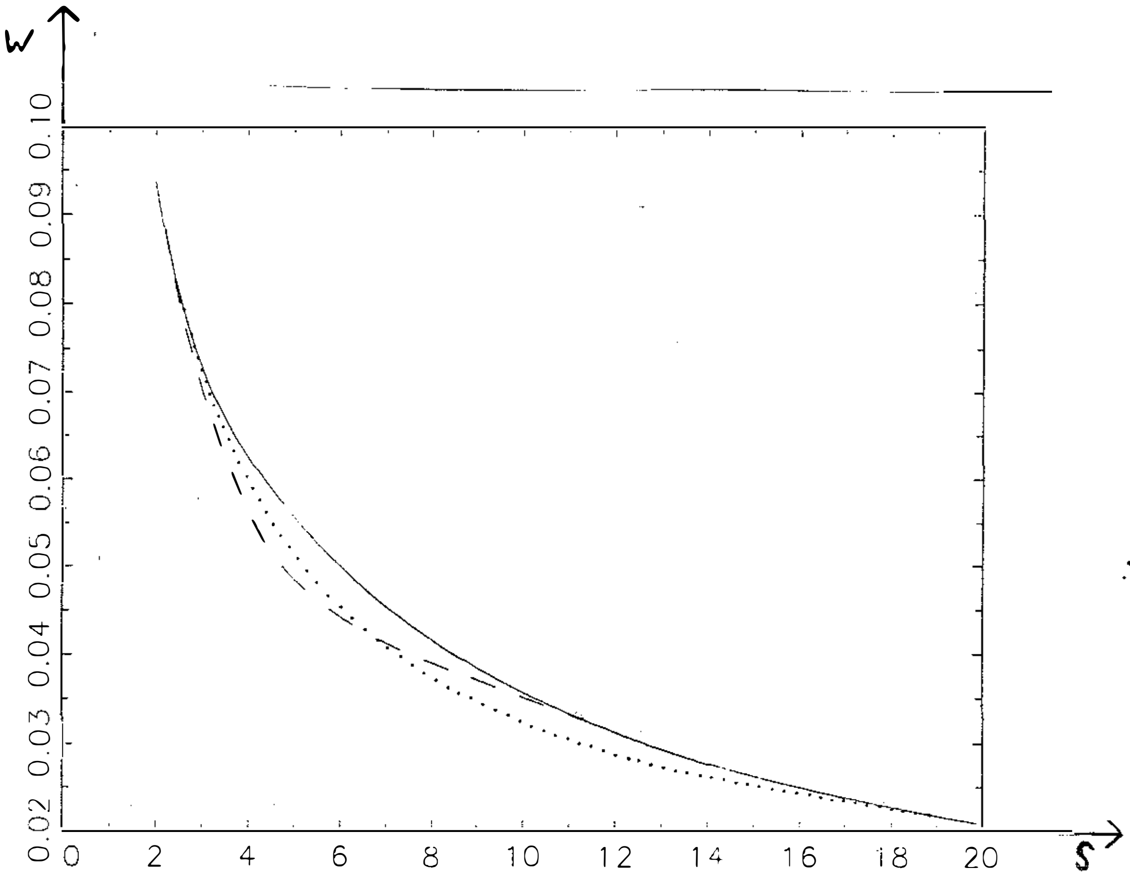

Figure 6: *Welfare without "lassez faire", solid line: social optimum, "big dashed" line: permits, dotted line: taxes.*

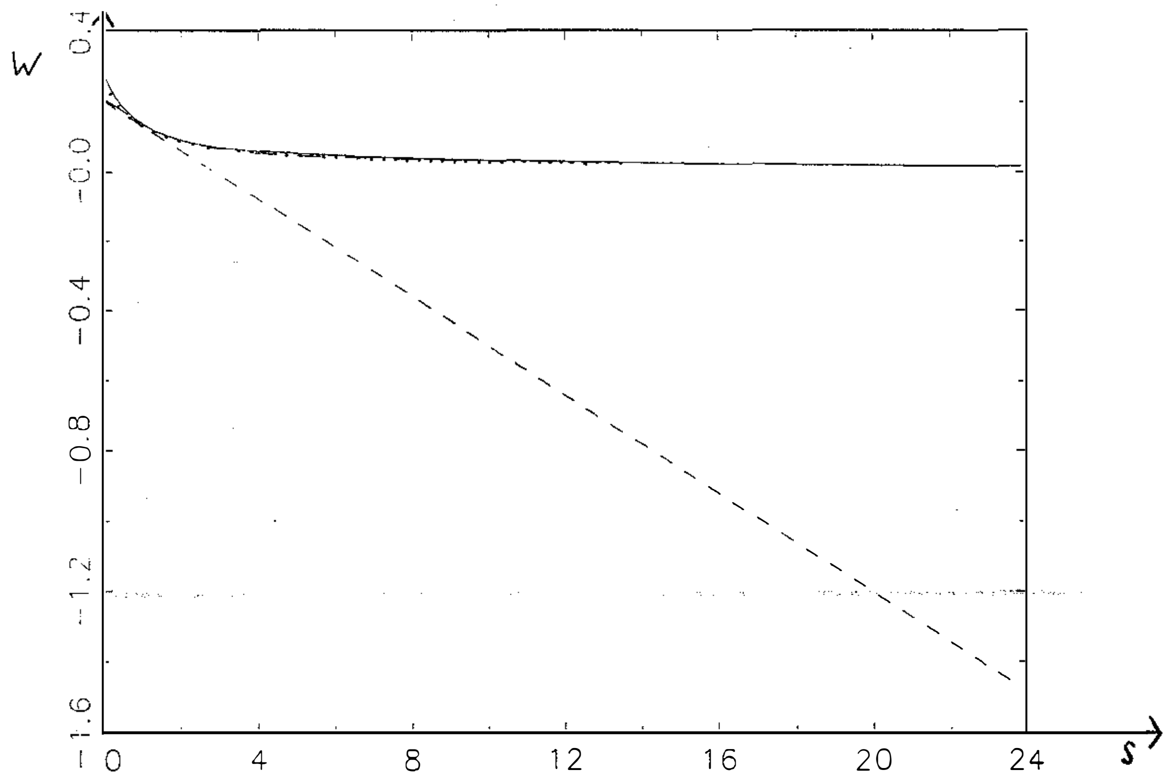

Figure 7: *Welfare including "lassez faire", solid line: social optimum, "big dashed" line: permits, dotted line: taxes, "small dashed" line: "laissez faire".*

References

[1] D. Baron and R. Meyerson. Regulating a monopolist with unknown costs. *Econometrica*, 50:911 – 930, 1982.

[2] W.J. Baumol and W.E. Oates. *The Theory of Environmental Policy.* Cambridge University Press, 1988. 2nd Ed.

[3] U. Ebert. On the effect of effluent fees under oligopoly: comparative statics. mimeo, 1991.

[4] U. Ebert. Pigouvian taxes and market structure: The case of oligopoly and different abatement technologies. *Finanzarchiv*, 1992. forthcoming.

[5] A. Endres. Do effluent charges (always) reduce environmental damage? *Oxford Economic Papers*, 37:255–261, 1985.

[6] R. Hahn. Market power and transferable property rights. *Quarterly Journal of Economics*, 99:753–765, 1984.

[7] E. Kwerel. To tell the truth: Imperfect information and optimal pollution control. *Rev. Econ. Stud.*, pages 595–601, 1977.

[8] D.A. Malueg. Welfare consequences of emission credit trading programs. *Journal of Environmental Economics and Management*, 18:66–77, 1990.

[9] E. Maskin and J. Riley. Monopoly with incomplete information. *The Rand Journal of Economics*, 15:171–196, 1982.

[10] W.D. Montgomery. Markets in licenses and efficient pollution control programs. *J. Econ. Th.*, 5:395–418, 1972.

[11] M.J. Roberts and M. Spence. Effluent charges and licenses under uncertainty. *Journal of Public Economics*, 5:193–208, 1990.

[12] M. Rothschild and J. Stiglitz. Equilibrium in competitive insurance markets. *Quarterly Journal of Economics*, 90:629–650, 1976.

[13] H. Siebert. Interview. *Der Spiegel*, 33:33–40, 1989.

[14] D. Spulber. Effluent regulation and long–run optimality. *Journal of Environmental Economics and Management*, 12:103–116, 1985.

[15] D. Spulber. Optimal environmental regulation under asymmetric information. *Journal of Public Economics*, 35:163–181, 1988.

[16] J. Stiglitz. Monopoly, nonlinear pricing and imperfect information: the insurance market. *Rev. Econ. Stud.*, 44:407–430, 1977.

[17] T.H. Tietenberg. *Emissions Trading: An Exercise in Reforming Pollution Policy.* Recources for the Future, Washington D.C., 1985.